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Non-perturbative orientifold transitions at the conifold

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Abstract: After orientifold projection, the conifold singularity in hypermultiplet moduli space of Calabi-Yau compactifications cannot be avoided by geometric deformations. We study the non-perturbative fate of this singularity in a local model involving O6-planes and D6-branes wrapping the deformed conifold in type-IIA string theory. We classify possible A-type orientifolds of the deformed conifold and find that they cannot all be continued to the small resolution. When passing through the singularity on the deformed side, the O-plane charge generally jumps by the class of the vanishing cycle. To decide which classical configurations are dynamically connected, we construct the quantum moduli space by lifting the orientifold to M-theory as well as by looking at the superpotential. We find a rich pattern of smooth and phase transitions depending on the total sixbrane charge. Non-BPS states from branes wrapped on non-supersymmetric bolts are responsible for a phase transition. We also clarify the nature of a $\mathbb{Z}_2$ valued D0-brane charge in the 6-brane background. Along the way, we obtain a new metric of G_2 holonomy corresponding to an O6-plane on the three sphere of the deformed conifold.

Keywords: Superstring Vacua, M-Theory, D-branes, Brane Dynamics in Gauge Theories.

Contents

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1. Introduction

The space of all string compactifications with $\mathcal{N}=1$ supersymmetry in four dimensions is expected to be quite rich. A poor man's approach to the problem is to think of $\mathcal{N}=1$ as $\mathcal{N}=2$ supersymmetry broken by branes, orientifolds, and fluxes. $\mathcal{N}=2$ compactifications have been extensively studied in the framework of type-II superstrings on Calabi-Yau manifolds, and we already have a picture of the variety of vacua. The most striking aspect of this picture is that the moduli spaces corresponding to different Calabi-Yau manifolds are connected to each other through conifold transitions [1, 2], which are interpreted as the $\mathcal{N}=2$ Higgs mechanism from the viewpoint of the four-dimensional spacetime physics.

Part of the motivation for our work actually comes from the desire to understand better the stringy interior of $\mathcal{N}=1$ Calabi-Yau "moduli" space, where we are putting quotation marks to emphasize that these moduli will generically be lifted by potentials. In particular, one would like to know which classical configurations are connected as parameters are varied, whether they are connected smoothly or through phase transitions, and what is the physical nature of the continuation or the transition. By a classical configuration, we mean the geometrical data of internal space, orientifold action, branes and fluxes in the large volume and weakly coupled regime where string and non-perturbative corrections are small. It is expected that the conifolds again play important roles.

There are various motivations to study conifolds in $\mathcal{N}=1$ compactifications. In the context of type-IIB flux compactifications [3], conifolds are the key to explore models with large hierarchy of scales [4]. Also, recent study shows that a good portion of flux vacua populates a neighborhood of conifolds [5, 6]. Conifolds are attractive also from cosmology, see for example [7, 8]. In this context, conifolds were approached from the "vector side" (complex structure in IIB, Kähler class in IIA). We would like to consider also approaching from the other, "hyper side" (complex structure in IIA, Kähler class in IIB). The approach to conifolds from hypermultiplet moduli space has been studied in $\mathcal{N}=2$ systems in [9].

In this paper, we shall study the local behavior of the $\mathcal{N}=1$ moduli space around the conifold loci, in type-II orientifolds. Before orientifolding, the conifold locus appears in real codimension two or more in $\mathcal{N}=2$ vector multiplet or hypermultiplet moduli spaces. The orientifold projects out a part of such closed string moduli fields (see [10] for details). In particular, it cuts a real slice in the classical geometric portion of the hypermultiplet moduli space, and the singularity appears in real codimension one. See the left hand side of figure 1. Since scalar fields in $\mathcal{N}=1$ chiral multiplets are complex, singularities should only be expected in complex co-dimension one. The issue is that the superpartners of

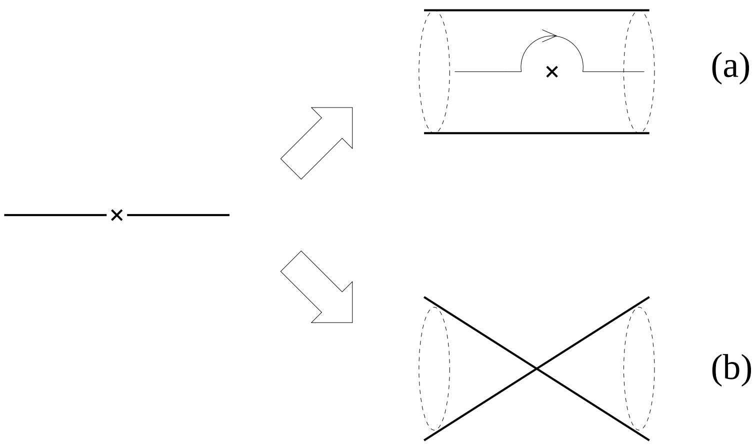

Figure 1: Is a μ-transition possible? When complexified with the RR field, the moduli space can be smooth or it can have a real codimension two singularity.

the geometric fields are Ramond-Ramond (RR) moduli, and the mixing between the two involves non-perturbative effects. It is therefore far from obvious what the complexified parameter space will look like in the vicinity of the classical singularity. Two possibilities are sketched on the right of figure 1.

For example, it was found in [11] that the tadpole cancellation conditions are generally different on the two sides of the conifold singularity in the real slice of Calabi-Yau moduli space. In other words, the charge of the orientifold plane is changing, and any claim of a smooth interpolation between the two classical limits has to account for this jumping charge.

While these points were raised for compact models, we will here focus on the singularity and study answers to these questions in the local model involving just the conifold. Although the results may not be directly applicable to compact models, we can test various methods to study the problem.

Since the conifold has been studied from a large number of perspectives in recent years, it might appear that answers to all these questions should be known. We therefore explain the basic point which we feel has not been addressed in full detail until now. Then we shall summarize our methods and results.

1.1 Basic question

Consider type-IIA orientifolds of the deformed conifold

$$\sum_{i=1}^{4} z_i^2 = \mu \,, \tag{1.1}$$

with respect to the anti-holomorphic involution

$$z_i \to \bar{z}_i \,. \tag{1.2}$$

Under this projection, the space of complex structures of the conifold, parameterized by μ, is restricted to the real slice $\mu \in \mathbb{R}$. The orientifold 6-plane, given by the fixed point set of (1.2) times flat four-dimensional Minkowski space is the locus of real solutions of (1.1). When $\mu > 0$, this leaves an S^3 worth, while if $\mu < 0$, there is no real solution, and the O-plane is empty. The point $\mu = 0$ is the classical conifold singularity. We will refer to the transition between μ positive and μ negative as the *μ-transition*. As we will see, whether or not the μ-transition is possible depends on the case.

As alluded to above, the real parameter μ is complexified by a RR field, which here arises as the period of the RR three-form around the vanishing S^3 of the deformed conifold. The fundamental question is whether or not this complexification allows the two classical branches ($\mu > 0$ and $\mu < 0$) to be connected in the full quantum theory. Evidently there are two more classical branches joining in at $\mu = 0$, the resolved conifolds, and the fate of these branches should be a part of the question.

More generally, we may choose to wrap D-branes on top of the O-plane on the vanishing S^3 of the deformed conifold. These will support an $\mathcal{N} = 1$ gauge theory at low energies, and one has to make sure that the proposed quantum dynamics takes account of the vacuum structure.

1.2 Lift to M-theory

Having exposed the problem, we now explain how we will address it. The main tool for us will be the lift to M-theory, as studied in [12−14], and many other places. Recall that D6-branes and O6-planes lift in M-theory to purely geometric configurations. In the local model, the essential idea is to identify all possible classical geometries with fixed asymptotics. It then turns out that with reasonable assumptions about the dynamics, holomorphy essentially completely determines the quantum moduli space relating the various geometries.

Our second method for analyzing the possible transitions, due to [15, 13] is to study the critical points of a certain superpotential W. The superpotential is computed on the branch of the resolved conifold, and in the orientifold situation is a combination of flux and crosscap contributions [16]. We give a careful analysis of this superpotential in section 5, and reproduce the component of the moduli space including the two resolved conifold points.

1.3 Main results

Consider wrapping N D6-branes on top of the O6-plane on the $S^3_>$ at the bottom of the orientifolded conifold (1.1), (1.2) for $\mu > 0$. Our convention is that the total 6-brane charge as measured at infinity is $2N - 4$ in the cover ($N - 2$ in the quotient), where -4 is the contribution from the O6-plane. Clearly, if a transition to $\mu < 0$ is possible, where there is no O-plane, the charge must be carried by $2N - 4$ D6-branes wrapped on the $S^3_<$ at the bottom of the conifold with $\mu < 0$. Now notice that a D6-brane wrapped on $S^3_>$ preserves the opposite combination of supersymmetries to a D6-brane wrapped on $S^3_<$. This is because the relevant calibration is the real part of the holomorphic three-form Ω, which when restricted to the two S^3's leads to opposite orientation, depending on whether μ is positive or negative. In other words, if we fix the supersymmetry preserved

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O-planes and D-branes wrapping the conifold			M-theory geometries		
blown-up flux thru $\mathbb{RP}^2$	deformed$(\mu > 0)$ gauge group on N D6-branes	deformed$(\mu < 0)$ $4-2N$ $\overline{\text{D6}}$-branes	blown-up	$\mu > 0$	$\mu < 0$
$N-2 \geq 3$	$\mathrm{SO}(2N)/\mathrm{Sp}(N-4)$	—	$\mathbb{D}_7/D_N$	$\mathbb{B}_7/D_N$	—
$N-2 = 3$	SO(10) or Sp(1)	—	$\mathbb{D}_7/D_5$	$\mathbb{B}_7/D_5$	—
$N-2 = 2$	SO(8) or none	—	$\mathbb{D}_7/D_4$	$\mathbb{B}_7/D_4$	—
$N-2 = 1$	SO(6)	—	$\mathbb{D}_7/D_3$	$\mathbb{B}_7/D_3$	—
$N-2 = 0$	SO(4)	none	(conifold $\times$ S^1)$/\mathbb{Z}_2$		
$N-2 = -1$	SO(2)	SO(2) or Sp(1)	$\mathbb{D}_7/D'_3$	$\mathbb{A}_7$	$\mathbb{B}_7/D'_3$
$N-2 = -2$	none	SO(4) or Sp(2)	$\mathbb{D}_7/D'_4$	$\mathbb{A}_7/\mathbb{Z}_2$	$\mathbb{B}_7/D'_4$
$N-2 = -3$	—	SO(6) or Sp(3)	$\mathbb{D}_7/D'_5$	—	$\mathbb{B}_7/D'_5$
$N-2 \leq -3$	—	$\mathrm{SO}(2(2-N))/\mathrm{Sp}(2-N)$	$\mathbb{D}_7/D'_{4-N}$	—	$\mathbb{B}_7/D'_{4-N}$

Table 1: M-theory geometries with various fixed values of Kaluza-Klein flux, their type-IIA interpretations, and low energy gauge group. The moduli spaces of supersymmetric vacua that link these various semi-classical limits can be contemplated in figure 16 on page 54.

at infinity, we can have D6-branes wrapped on $S^3_>$ or anti-D6-branes[1] wrapped on $S^3_<$. It is clear, therefore, that we can at best expect a transition between $\mu > 0$ and $\mu < 0$ in the supersymmetric parameter space if N is non-negative (so that the D6-branes preserve the same susy as the O6-plane), and $2N-4$ is non-positive (so that to conserve the charge, we wrap $4-2N$ anti-D6-branes). In other words, we can expect a μ-transition if $N = 0$, 1, or 2.

In the previous paragraph, we discussed the possibility of wrapping an $\mathrm{O6}^-$-plane on $S^3_>$ with N D6-branes, which yields an $\mathrm{SO}(2N)$ gauge group. Alternatively, for $N \geq 4$ we may wrap an $\mathrm{O6}^+$ and $N-4$ D6-branes, yielding an $\mathrm{Sp}(N-4)$ gauge group. For $N < 1$ there is a similar choice for the action of the free orientifold on the Chan-Paton matrices corresponding to the D6-branes on $S^3_<$. This leads to two distinct possibilities for the low energy four-dimensional gauge group, $\mathrm{SO}(2(2-N))$ or $\mathrm{Sp}(2-N)$. Finally, for any value of N we have two semi-classical limits corresponding to the two resolved conifolds with freely acting orientifold and $N-2$ units of RR 2-from flux through $\mathbb{RP}^2$.

With these observations in mind, we can identify all the possible semi-classical limits in the IIA description for each value of N. Our results are summarized in the left-hand side of table 1.

The M-theory lifts of the various semi-classical limits are described in the right-hand side of table 1. The problem for $N \underset{(}{\geqq}_{)} 3$ is included in [12, 14]. For infinite string coupling (the size of the M-theory circle growing without bounds asymptotically), the M-theory geometries are quotients of smooth G_2 holonomy manifolds, X_i, by the dihedral group D_N.

[1] Here and throughout the paper, we refer to objects as branes or anti-branes according to the sign of the charge measured at infinity. A more natural convention would be to refer to all objects preserving the same supersymmetry as branes, but we choose our present convention to emphasize that the supersymmetric objects sometimes carry opposite charge.

The X_i are all isomorphic to the spin bundle over S^3, whose G_2 holonomy metric was found in [17]. They differ in the breaking pattern of the asymptotic discrete symmetries, and discrete fluxes at the singularities. In the D_N case, there are four semi-classical limits corresponding to the four IIA limits described above.

For $N = 3$, the dihedral group D_3 is isomorphic to $A_3 = \mathbb{Z}_4$, and the problem is equivalent, from the M-theory perspective, to a case without orientifold. But for the appropriate identification of the M-theory circle, we still end up with an orientifold in type IIA.

The extension to lower values of N, as well as the generalization to finite values of the string coupling, requires more complicated G_2 holonomy metrics with reduced symmetry. These metrics have been partially constructed in [18 − 20], and we now describe their relevance to our problem.

For $N > 2$, if one wishes to keep the asymptotic IIA string coupling finite, the M-theory lift of the deformed conifold geometry involves a G_2 metric called $\mathbb{B}_7$ in [19]. Roughly, the manifold $\mathbb{B}_7$ is a Taub-NUT manifold fibered over an S^3. Quotienting $\mathbb{B}_7$ by the dihedral group leaves a D_N singularity supporting an SO($2N$) or Sp($N-4$) gauge group, depending on a discrete flux. The resolved conifold with flux (and finite string coupling) has an M-theory lift called $\mathbb{D}_7$ in [19]. This G_2 metric is smooth and D_N acts freely on $\mathbb{D}_7$.

For $N = 2$, the M-theory lift will be

$$\frac{(\text{conifold}) \times S^1}{\mathbb{Z}_2},$$

where $\mathbb{Z}_2$ acts as an antiholomorphic involution on the conifold and reverses the M-theory S^1. As such, we know the classical M-theory geometry exactly.

For $N = 0$ or $N = 1$, one can guess that the M-theory lift of the deformed conifold with O-plane (namely, $\mu > 0$) will look like an Atiyah-Hitchin manifold [21] or Dancer's manifold [22] fibered over an S^3. Such G_2 metrics were not previously known but we find that they do indeed exist. We will call these manifolds $\mathbb{A}_7/\mathbb{Z}_2$ and $\mathbb{A}_7$, respectively.

In all cases with $N < 2$, the deformed conifold with $\mu < 0$ (which is wrapped by $4-2N$ anti-D6-branes) and the resolved conifold with flux again lift to the quotient of $\mathbb{B}_7$ and $\mathbb{D}_7$, respectively, by the dihedral group D_{4-N}. The difference to the $N > 2$ case is in the action of D_{4-N} on the space, as we will explain in more detail later. The gauge group living on the $4-2N$ anti-D6-branes in the freely acting orientifold can be SO($2(2-N)$) or Sp($2-N$), depending on the action on the Chan-Paton factors. This corresponds to the value of a discrete torsion. Thus, the $\mathbb{B}_7/D_{4-N}$ geometries yield two semi-classical limits with $\mu < 0$ in each case.

No analytic expressions are known for any of the G_2 metrics $\mathbb{A}_7$, $\mathbb{B}_7$, $\mathbb{D}_7$ except one special point on the parameter space of $\mathbb{B}_7$ found in [18], as well as the limits of zero or infinite string coupling. From symmetry requirements, one can determine the metrics up to a small number of unknown functions (of a radial coordinate) and derive a set of differential equations for the unknowns. It is not difficult to verify numerically the existence of such solutions. One can also see that they depend on two parameters, one corresponding to the radius of the M-theory circle at infinity and the other to the volume of S^3 at the center.

The next question is how these semi-classical limits with the same asymptotic flux, $N-2$, and the same supersymmetry fit together into a complex space parameterizing supersymmetric vacua. For $N \mathrel{_{(}{\geqq}_{)}} 3$, which is the case discussed in [14], the parameter space is a copy of $\mathbb{P}^1$ with four (three for $N=3$) marked points corresponding to the semi-classical limits we have discussed above. Good local coordinates around each of these points correspond to the volume deficits of certain three-cycles in the asymptotic geometries together with the period of the M-theory three-form around the same three-cycles. These complex parameters are the instanton coefficients in the four-dimensional low-energy gauge theory.

The case $N=2$ was discussed in [23], where it was argued that a μ-transition should be possible. Our analysis will confirm the expectation in this case. In addition, we will find a second branch of moduli space containing two vacua of SO(4) gauge theory as well as the two resolved conifolds that were not treated in [23].

In the case $N=1$, there are five semi-classical limits. As can be seen from table 1 two of them have an $\mathrm{SO}(2) \cong \mathrm{U}(1)$ gauge theory at low energies, one has an $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$, while the two remaining ones have no gauge theory at all. The first two have free massless gauge bosons while the latter three points do not. Since the massless spectrum is different, one must pass through a phase transition when interpolating between the various limits. This situation is very similar to ones studied recently in [24], and we will be able to use these methods to deduce the structure of the quantum parameter space, confirming the naive picture we have just sketched.

When $N=0$, it appears at first sight that we have also five semi-classical limits: two on the resolved conifold, two from the deformed conifold with $\mu < 0$, and one from the deformed conifold with $\mu > 0$. However, as we will see, there are in fact two distinct semi-classical limits corresponding to just the O-plane wrapping the S^3. In M-theory, this can be simply seen from the existence of an asymptotic discrete $\mathbb{Z}_2'$ symmetry that is spontaneously broken in the interior of $\mathbb{A}_7/\mathbb{Z}_2$. In type-IIA string theory, this symmetry corresponds to D0-brane charge modulo 2, which is broken only at the bare orientifold plane, but is preserved in the presence of just a single D6-brane on top. (We will explain why this statement is not in conflict with the K-theory classification of D-brane charge.) There are therefore six semi-classical limits to consider. We will argue that the quantum parameter space consists of two disconnected branches, with a certain distribution of vacua consistent with the discrete symmetries of the problem.

For $N < 0$, we again have four semi-classical limits, each of which has a mass gap at low energies. From the analysis of the holomorphic parameters associated with the gauge theories, as well as our later superpotential analysis, we will deduce that the space on which these four limits sit is again a copy of $\mathbb{P}^1$. In fact, we will see that the curve for $N < 0$ is isomorphic to the curve for $N' = 4 - N > 4$, with the only difference being the association between two of the points and SO / Sp gauge group!

1.4 Summary

In addition to the literature that we have cited already, aspects of the problem have also been discussed elsewhere. The basic question whose solution we have presented in the

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previous subsection has been broached, for example, in [23, 25], with a restriction to the locally tadpole canceling case with exactly two D6-branes on top of the O-plane. More recently, similar transitions have been found in type-IIB compactifications in [26]. The main result of our paper is to explain under which conditions in the local model we can actually expect a transition, and to determine the quantum parameter space, whenever possible. Our main tool of analysis is the lift to M-theory on G_2 manifolds which are described as quotients by finite groups. Similar system have been analyzed in the past also in [27], for example. Conifold transitions in the G_2 context have also been studied in [28].

We also have a number of subsidiary and complementary results to offer, as we now summarize. We will start out with a classification of A-type orientifolds[2] of the deformed conifold in section 2. In this section, we also show that when passing through a "μ-transition" between $\mu > 0$ and $\mu < 0$ (independent of whether this is possible dynamically or not), the orientifold charge changes by the class of the vanishing cycle. We also study how the previous orientifolds act on the resolved conifold. We then return to the main case of interest, the orientifold (1.2). In section 3, we explain generalities about the symmetries of the underlying G_2 holonomy metrics. The nuts and bolts of these space are assembled in section 4. We find a new class of G_2 holonomy metrics which we label $\mathbb{A}_7$. In section 5, we derive the quantum parameter spaces. In particular, we show how a careful analysis of the Vafa superpotential produces most of the structure of the parameter space. For the cases $N = 2, 1$ and 0, we need to invoke some additional information, partly from [24]. In section 7, we briefly discuss the problem of μ-transition for the other classes of orientifolds. Since they break more of the geometrical symmetries, we are unable to write down explicit metrics. The Vafa superpotential gives a prediction for one other class of models. However, when the orientifold does not admit the resolved conifold, the superpotential method is not applicable. In these cases, we describe our best educated guesses for possible μ-transitions. We finally conclude in section 8.

2. Orientifolds of the conifold

In this section, we describe the possible A-type orientifolds of the deformed conifold. In particular, we will see that when crossing the conifold the class of the orientifold locus changes by the class of the vanishing cycle. We then analyze how these orientifolds act on the resolved conifold.

2.1 Deformed conifold

An orientifold of the deformed conifold

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = \mu \tag{2.1}$$

in type-IIA string theory can be obtained from an anti-holomorphic involution, which acts on the complex coordinates z_i by complex conjugation followed by a symmetry of the

[2] The construction of several orientifolds of the conifold in type IIB has been analyzed in [29]. These orientifolds were constructed by partially blowing up orientifolds of orbifold singularities.

quadric (2.1),

$$z_i \to e^{i\alpha} M_i^j \bar{z}_j . \tag{2.2}$$

Here M is an orthogonal matrix with $M^2 = 1$, and α is a phase, which if we assume that μ is real can be set to zero.

Since all such orthogonal matrices can be diagonalized with ± 1 on the diagonal, inequivalent anti-holomorphic involutions of the deformed conifold are classified by the number of $+1$ and -1 eigenvalues of M.

In each of these five possibilities, the fixed point set is described by setting $z_i = x_i$ or $z_i = i y_i$, with x_i, y_i real, depending on the corresponding sign. The fixed point locus are the orientifold 6-planes. Let us take $\mu > 0$ (the case with μ negative can easily be obtained by interchanging the real and imaginary components of the z_i). We have the following inequivalent cases:

(0) For $(z_1, z_2, z_3, z_4) \to (\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4)$ the O6-plane is described by

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = \mu . \tag{2.3}$$

Since $\mu > 0$, this is an S^3.

(1) If the involution takes $(z_1, z_2, z_3, z_4) \to (-\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4)$ the orientifold set is non-compact,

$$-y_1^2 + x_2^2 + x_3^2 + x_4^2 = \mu , \tag{2.4}$$

and isomorphic to $S^2 \times \mathbb{R}$.

(2) When $(z_1, z_2, z_3, z_4) \to (-\bar{z}_1, -\bar{z}_2, \bar{z}_3, \bar{z}_4)$ the O-plane is at

$$-y_1^2 - y_2^2 + y_3^2 + y_4^2 = \mu , \tag{2.5}$$

which describes $S^1 \times \mathbb{R}^2$.

(3) If $(z_1, z_2, z_3, z_4) \to (-\bar{z}_1, -\bar{z}_2, -\bar{z}_3, \bar{z}_4)$ the orientifold set is not connected: The equation

$$-y_1^2 - y_2^2 - y_3^2 + x_4^2 = \mu \tag{2.6}$$

is solved by two copies of $\mathbb{R}^3$.

(4) Finally, when $(z_1, z_2, z_3, z_4) \to (-\bar{z}_1, -\bar{z}_2, -\bar{z}_3, -\bar{z}_4)$, the orientifold set is empty:

$$-y_1^2 - y_2^2 - y_3^2 - y_4^2 = \mu \tag{2.7}$$

has no real solutions.

Generalizing the relation between (0) and (4) discussed in the introduction, flipping the sign of μ maps (1) to (3) and (2) to itself.

We can introduce D6-branes wrapping supersymmetric cycles of the deformed conifold. In order to have dynamical gauge symmetry in four dimensions, the cycle must be compact, and the only possibility is the three-dimensional sphere at the center of the conifold. In

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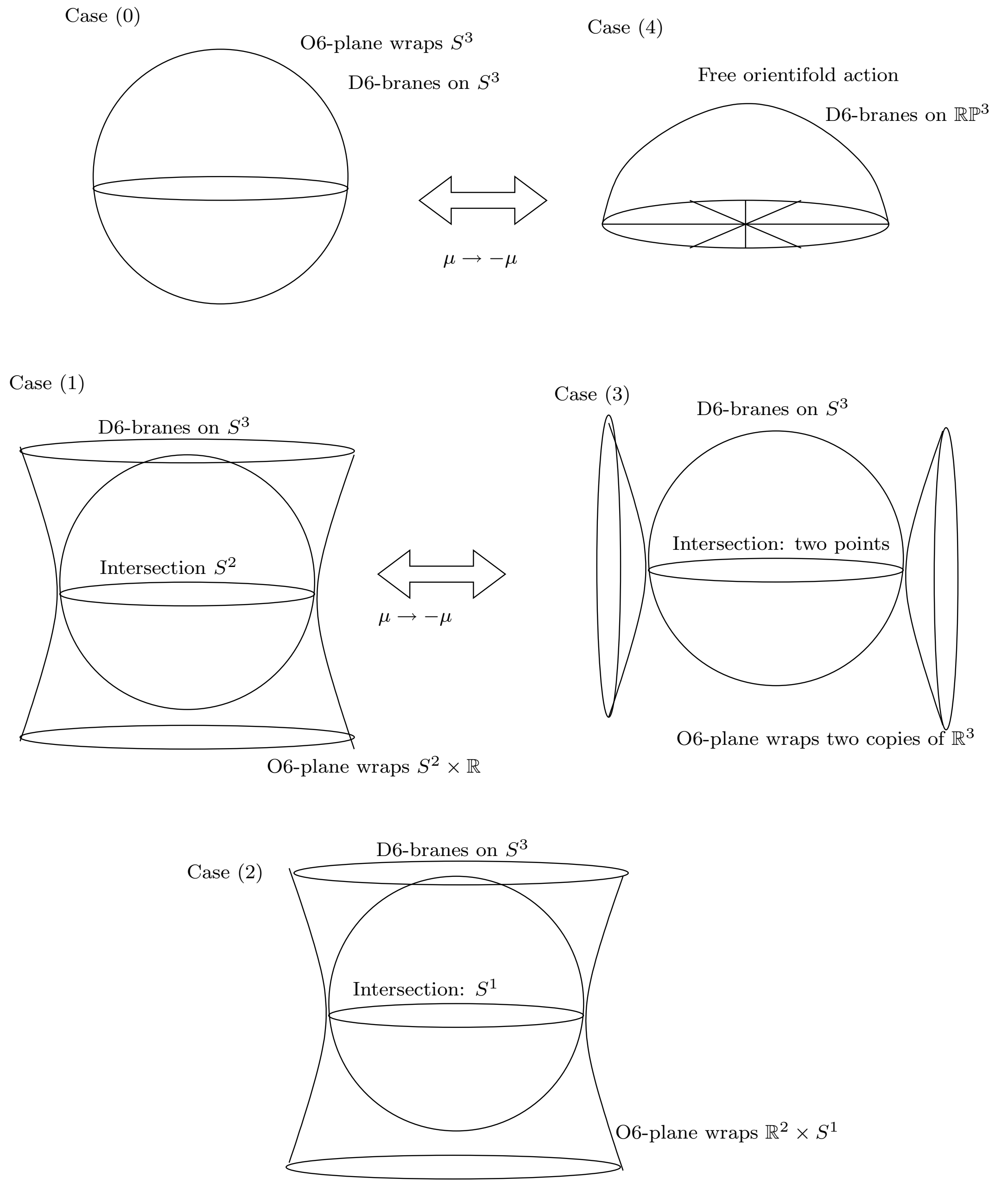

Figure 2: Representations of the 5 orientifolds of the deformed conifold in type-IIA theory. The orientifold loci are the fixed points of the anti-holomorphic involution. The cases (1) and (3) break supersymmetry when D6-branes are wrapped on S^3 and they are related by taking $\mu \to -\mu$.

the cases (1) and (3), the involution reverses the orientation of the S^3, i.e., maps branes to anti-branes. Therefore, wrapping branes on S^3 in these cases will break supersymmetry. In fact, in those cases, there is a RR tadpole which originates from the non-trivial intersection of the O-plane with the compact S^3. To get a stable configuration, we are forced to wrap branes on some other non-compact cycles. This yields quite an interesting class of models, which is briefly analyzed in section 7. The case (0) is supersymmetric: the D6-branes wrap the same S^3 as the orientifold. In the case (4) the action of the orientifold is free,

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the D6-branes are then wrapping an $\mathbb{RP}^3$. In case (2) the orientifold plane intersects the S^3 where the D6-branes are wrapping in an S^1. We have depicted these circumstances in figure 2.

Compact orientifold models including invariant conifolds must be one of these five types. For example, let us consider the type-IIA orientifold on the mirror X of the Fermat quintic, which is a resolution of the orbifold of

$$z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 - 5\psi z_1 z_2 z_3 z_4 z_5 = 0$$

by the $\mathbb{Z}_5^3$ action, $z_i \to \omega_i z_i$, $\omega_i^5 = \prod \omega_i = 1$. For $\psi = 1$, X has a single conifold singularity at $z_1 = \cdots = z_5 = 1$. We consider the orientifold with respect to the involution $\tau_\sigma : z_i \to \overline{z}_{\sigma(i)}$, where σ is an order two permutation (exchange), that fixes the conifold singularity when $\psi = 1$. The orientifold is allowed only when ψ is real. There are three distinct cases — σ is identity, an exchange of a pair, and an exchange of two pairs. Depending on the sign of $\epsilon = \psi - 1$, these cases are one of the five possibilities: Without exchange, $z_i \to \overline{z}_i$, it is case (0) if $\epsilon > 0$ and case (4) if $\epsilon < 0$. With an exchange of one pair, such as $(z_1, z_2, z_3, z_4, z_5) \to (\overline{z}_2, \overline{z}_1, \overline{z}_3, \overline{z}_4, \overline{z}_5)$, it is case (1) if $\epsilon > 0$ and case (3) if $\epsilon < 0$. With an exchange of two pairs, such as $(z_1, z_2, z_3, z_4, z_5) \to (\overline{z}_2, \overline{z}_1, \overline{z}_4, \overline{z}_3, \overline{z}_5)$, it is case (2) for both signs of ϵ. Note that cases (0) and (4) are mirror to type-IIB orientifold with an O9-plane (type I), case (1) and (3) are mirror to IIB orientifold with O3/O7 planes, and case (2) is mirror to IIB orientifold with O5-planes [10].

2.2 Homology of O-planes

We can study the homology classes of the lagrangian manifolds of the orientifold loci by computing the integral of the holomorphic three-form Ω around them, where

$$\Omega = \frac{dz_1 \wedge dz_2 \wedge dz_3}{z_4} \tag{2.8}$$

on the sheet $z_4 = \sqrt{\mu - z_1^2 - z_2^2 - z_3^2}$.

We find that

(0) For the case (0) the orientifold wraps a compact manifold with period:

$$\varpi^{(0)}(\mu) = \int_{S^3} \Omega = 2 \int_{x_1^2+x_2^2+x_3^2 \leq \mu} \frac{dx_1 dx_2 dx_3}{\sqrt{\mu - x_1^2 - x_2^2 - x_3^2}} = 8\pi\mu \int_0^1 \frac{r^2 dr}{\sqrt{1-r^2}} = 2\pi^2 \mu\,. \tag{2.9}$$

(1) The orientifold locus is non-compact, and we need to introduce a cutoff, which we put at $|z|^2 = x^2 + y^2 = \Lambda$.

$$\begin{aligned} \varpi^{(1)}(\mu) &= \int_{S^2 \times \mathbb{R}} \Omega = 2i \int_{\mu \leq x_2^2 + x_3^2 + x_4^2 \leq \frac{\Lambda+\mu}{2}} \frac{dx_2 dx_3 dx_4}{\sqrt{x_2^2 + x_3^2 + x_3^2 - \mu}} \\ &= 8\pi i \mu \int_1^{\sqrt{\frac{\Lambda+\mu}{2\mu}}} \frac{r^2 dr}{\sqrt{r^2 - 1}} = 4\pi i \mu \left[\frac{\sqrt{\Lambda^2 - \mu^2}}{2\mu} + \log \frac{\sqrt{\Lambda + \mu} + \sqrt{\Lambda - \mu}}{\sqrt{2\mu}} \right]. \end{aligned} \tag{2.10}$$

(2) As in the previous case, the orientifold is non-compact.

$$
\begin{aligned}
\varpi^{(2)}(\mu) = \int_{S^1\times\mathbb{R}^2} \Omega = -2 \int_{\substack{\mu+y_1^2+y_2^2-x_3^2\geq 0\\ y_1^2+y_2^2\leq\frac{\Lambda-\mu}{2}}} \frac{dy_1 dy_2 dx_3}{\sqrt{\mu+y_1^2+y_2^2-x_3^2}} \\
= -\pi^2(\Lambda-\mu)\,.
\end{aligned}
\tag{2.11}
$$

(3) We have:

$$
\varpi^{(3)}(\mu) = \int_{\mathbb{R}^3} \Omega = -2i \int_{y_1^2+y_2^2+y_3^2\leq\frac{\Lambda-\mu}{2}} \frac{dy_1 dy_2 dy_3}{\sqrt{\mu+y_1^2+y_2^2+y_3^2}} \tag{2.12}
$$

$$
= -8\pi i\mu \int_0^{\sqrt{\frac{\Lambda-\mu}{2\mu}}} \frac{r^2 dr}{\sqrt{1+r^2}} = -4\pi i\mu\left[\frac{\sqrt{\Lambda^2-\mu^2}}{2\mu} - \log\frac{\sqrt{\Lambda+\mu}+\sqrt{\Lambda-\mu}}{\sqrt{2\mu}}\right].
$$

(4) This is empty.

If we evaluate case (1) for $\mu < 0$ (following the computation in case (3)), and subtract the result from the direct analytic continuation of (1), the difference is

$$
\varpi^{(1)}(\mu) - \varpi^{(1)}(-\mu) = 4\pi i\mu\log\sqrt{-1} = 2\pi^2\mu, \tag{2.13}
$$

which is exactly the period of the vanishing S^3. The same result also holds for case (2)

$$
\varpi^{(2)}(\mu) - \varpi^{(2)}(-\mu) = -\pi^2(\Lambda-\mu) - (-\pi^2(\Lambda+\mu)) = 2\pi^2\mu\,, \tag{2.14}
$$

and trivially for (0)/(4).

We can understand this by noting that the transition does not affect the boundary of the O-plane and so we may glue the O-planes for $\mu < 0$ and $\mu > 0$ along their common boundary to form a compact three-cycle which must then be homologous to an integer multiple of the minimal S^3. The calculation shows that this integer is one.

It is natural to propose that this holds as a universal result, and not just for the simple conifold singularity we have studied here. We conjecture:

> *When crossing the conifold locus of real co-dimension one in the geometric moduli space of an orientifold model, the class of the O-plane changes by the class of the vanishing cycle.*

The non-compact part of the homology of the O-planes (which contributes the $\mu\log\mu$ part in the expressions above) can be understood from the intersection with the compact three-cycle. It is easy to see that when we make the intersection between O-plane and the S^3 transversal, the S^1 in case (2) disappears completely, while the S^2 leaves two intersection points. This follows from the fact that for a lagrangian submanifold such as our three-cycles, the normal bundle is isomorphic to the tangent bundle via contraction with the Kähler form. The number of intersection points is then simply the Euler characteristic of the (non-transversal) intersection locus.

2.3 Orientifolds of the resolved conifold

Let us now discover what these involutions look like for the blown-up conifold. This space, also known as $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, can be written as

$$\begin{pmatrix} x & u \\ v & y \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = 0\,, \tag{2.15}$$

with $(\lambda_1, \lambda_2) \in \mathbb{P}^1$ and

$$\begin{aligned} x &= z_1 + i z_2\,, \qquad & y &= z_1 - i z_2\,, \\ u &= z_3 + i z_4\,, \qquad & v &= -z_3 + i z_4\,. \end{aligned} \tag{2.16}$$

The blow up breaks the O(4) symmetry we have used above to SO(4), while at the same time restoring the U(1) phase symmetry. Thus, in (2.2), M must have determinant $+1$, while α is a priori arbitrary. But note that by a change of coordinates, α can be conjugated to 0. The condition that the number of -1 eigenvalues of M must be even eliminates cases (1) and (3) discussed previously for the deformed conifold, while (0) and (4) become equivalent on the blown-up side. In cases (1) and (3) the Kähler parameter is projected out by the orientifold action and it is not possible to blow up the singularity. This leaves two cases:

(0) (equivalent to (4)) maps $(x, y, u, v) \to (\bar{y}, \bar{x}, -\bar{v}, -\bar{u})$. In terms of the inhomogeneous coordinate $z = \lambda_1/\lambda_2 = -u/x = -y/v$ the $\mathbb{P}^1$ is mapped as

$$z \to -\frac{1}{\bar{z}}\,. \tag{2.17}$$

This is a freely acting orientifold.

(2) The action $(x, y, u, v) \to (-\bar{y}, -\bar{x}, -\bar{v}, -\bar{u})$ must be accompanied by

$$z \to \frac{1}{\bar{z}}\,. \tag{2.18}$$

The fixed point set is an $S^1 \times \mathbb{R}^2$.

We note that these cases coincide with the cases where the orientifold action preserves the same supersymmetry as the D6-branes wrapping the S^3, and are precisely the cases discussed by Acharya, Aganagic, Hori and Vafa in [16].

2.4 The gauge group

For future reference, it is useful to describe here which gauge theories will be living on the worldvolume of D6-branes that are wrapping this geometry.

In flat space N dynamical D6-branes on the top of an O6$^-$-plane yield an SO($2N$) gauge group on the worldvolume. The O6$^-$-plane has a RR charge -2 in D6-brane units, so the total charge of the system is $N-2$. When wrapping an S^3 the D6-branes cannot be higgsed away. The four-dimensional gauge theory is pure $\mathcal{N}=1$ SO($2N$) super Yang-Mills. This theory is confining with $h = 2N-2$ different vacua.

A similar classical configuration is a system of $N-4$ D6-branes on the top of an O6$^+$-plane. The O6$^+$-plane has RR charge $+2$, so the whole system also has charge $N-2$. The low energy theory is pure $\mathcal{N}=1$ Sp$(N-4)$ super Yang-Mills. The theory is confining and has $h=N-3$ different vacua.

This discussion was of course standard. Slightly less familiar are D-branes wrapping in freely acting orientifolds (such as case (4) above), but it is also clear what will result. The involution acts on the S^3 as the antipodal map, $x\to -x$. Locally, this simply identifies excitations at antipodal points on the sphere, via an anti-unitary transformation on the D6-brane degrees of freedom. That can be understood as an action on the Chan-Paton matrices:

$$\lambda(x)\to -\gamma_\Omega\lambda(-x)^T\gamma_\Omega^{-1}\,, \tag{2.19}$$

where λ are $2M\times 2M$ hermitian matrices for $2M$ D6-branes wrapping the covering $S^3\ni x$. Locally on S^3, this simply yields a U$(2M)$ gauge group. The zero modes, however, suffer a slightly different projection. The orientifold action is an involution if $\gamma_\Omega(\gamma_\Omega^T)^{-1}=\epsilon$, with $\epsilon=\pm 1$, i.e. γ_Ω is symmetric or antisymmetric. Depending on the sign the four-dimensional theory will be pure super Yang-Mills with gauge group SO$(2M)$ or Sp(M). As before the system preserves $\mathcal{N}=1$.

3. Symmetries of G_2 holonomy metrics

In this section and the next, we discuss aspects of the G_2 lift of the deformed and resolved conifold with branes and fluxes. Most of this section is review [30, 18, 19], but the careful discussion of the symmetry breaking pattern will be crucial in our subsequent analysis.

3.1 Deformed and resolved conifold

We begin by recording the symmetry group of the conical Calabi-Yau metric on the (singular) conifold,

$$z_1^2+z_2^2+z_3^2+z_4^2=0\,. \tag{3.1}$$

These isometries must preserve (3.1) together with

$$r^2=\sum|z_i|^2\,. \tag{3.2}$$

In full glory, the symmetry group is

$$\big(\mathrm{SU}(2)\times\mathrm{SU}(2)\ltimes\widetilde{\mathbb{Z}}_2\times\mathrm{U}(1)^{\mathrm{phase}}\ltimes\mathbb{Z}_2^{\mathrm{cc}}\big)/_{\mathbb{Z}_2\times\mathbb{Z}_2}\,. \tag{3.3}$$

Here, the SO(4) $\cong$ SU(2) $\times$ SU(2)$/\mathbb{Z}_2$ is extended by $\widetilde{\mathbb{Z}}_2$ to the O(4) leaving the quadric invariant. $\widetilde{\mathbb{Z}}_2$ acts as

$$(z_1,z_2,z_3,z_4)\mapsto(-z_1,z_2,z_3,z_4)\,.$$

The U(1)$^{\mathrm{phase}}$ contains the rotations

$$R_\alpha : z_i\mapsto e^{i\alpha/2}z_i$$

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with $\alpha \in [0, 4\pi]$, and $\alpha = 2\pi$ corresponding to an element of SO(4). Finally, $\mathbb{Z}_2^{\rm cc}$ is complex conjugation, represented by

$$c_0 : z_i \mapsto \bar{z}_i \,. \tag{3.4}$$

When conjugated by elements of $\mathrm{U}(1)^{\rm phase}$, c_0 becomes

$$c_\alpha = R_\alpha c_0 R_\alpha^{-1} : z_i \mapsto e^{i\alpha} \bar{z}_i \,. \tag{3.5}$$

When the singular conifold is smoothed out, some of these symmetries are broken. The *deformation* which replaces (3.1) by

$$\sum z_i^2 = \mu \tag{3.6}$$

breaks $\mathrm{U}(1)^{\rm phase}$ to the $\mathbb{Z}_2$ which is already part of O(4) (namely $\alpha = 2\pi$). Since (3.5) is a symmetry of (3.6) for both $\alpha = \arg(\mu)$ and $\alpha = \arg(\mu) + \pi$, we have a choice of orientifold $c_0 : z_i \mapsto \bar{z}_i$ or $c_\pi : z_i \mapsto -\bar{z}_i$.

The blowup of the conifold is obtained by rewriting (3.1) as

$$xy - uv = 0 \,, \tag{3.7}$$

and then replacing it with the two equations

$$\begin{pmatrix} x & u \\ v & y \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = 0 \tag{3.8}$$

in $\mathbb{C}^4 \times \mathbb{P}^1$. The blowup clearly preserves $\mathrm{U}(1)^{\rm phase}$, but breaks $\widetilde{\mathbb{Z}}_2$. Indeed, transposition of z amounts to exchanging u and v and is equivalent to flopping the $\mathbb{P}^1$.

As equation (3.5) shows, orientifolding breaks $\mathrm{U}(1)^{\rm phase}$ down to the $\mathbb{Z}_2$ subgroup which is in SO(4). The addition of RR 2-form flux also breaks this $\mathrm{U}(1)^{\rm phase}$ as will become clear after lifting to M-theory. Thus the symmetry group of the asymptotic IIA geometry relevant to our problem is:

$$\big(\mathrm{SO}(4) \ltimes \widetilde{\mathbb{Z}}_2 \times \mathbb{Z}_2^{\rm cc}\big). \tag{3.9}$$

Deforming the conifold leaves these symmetries intact whilst resolving breaks $\widetilde{\mathbb{Z}}_2$.

3.2 Lift

Next we would like to lift to M-theory and understand the action of the symmetry group (3.9) there. The asymptotic boundary of the conifold is $T^{1,1} = \mathrm{SU}(2) \times \mathrm{SU}(2)/\,\mathrm{U}(1)$ and in the presence of RR 2-form flux the boundary of the M-theory lift is an orbifold of $\mathrm{SU}(2) \times \mathrm{SU}(2)$.

This can be described a little more explicitly by introducing some coordinates on the conifold. We write quaternionically

$$z = \begin{pmatrix} z_1 + iz_2 & z_3 + iz_4 \\ -z_3 + iz_4 & z_1 - iz_2 \end{pmatrix} = x + iy \,, \tag{3.10}$$

with

$$
\begin{aligned}
x &= \begin{pmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{pmatrix} = X\tilde{X}^\dagger \,, \\
y &= \begin{pmatrix} y_1 + iy_2 & y_3 + iy_4 \\ -y_3 + iy_4 & y_1 - iy_2 \end{pmatrix} = X\sigma\tilde{X}^\dagger
\end{aligned} \tag{3.11}
$$

and $\sigma = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$. Here we have used that (3.1) and (3.2) imply that (for $r = \sqrt{2}$) x, y are orthogonal unit quaternions

$$
\det x = \det y = 1 \qquad \mathrm{Tr}\ y^\dagger x = 0\,. \tag{3.12}
$$

Choosing our standard traceless SU(2) matrix σ, this equation is solved in terms of two SU(2) matrices X and $\tilde{X}$, modulo the relation $(X, \tilde{X}) \equiv (X\Theta, \tilde{X}\Theta)$ with $\Theta = \left(\begin{smallmatrix} e^{i\psi/2} & 0 \\ 0 & e^{-i\psi/2} \end{smallmatrix}\right)$. This exhibits the base of the conifold as either a (topologically trivial) S^2 bundle over S^3 or as $T^{1,1} = \mathrm{SU}(2) \times \mathrm{SU}(2)/\,\mathrm{U}(1)$. (We won't need its presentation as $T^{1,0}$, which uses x and X instead.)

X and $\tilde{X}$ are the $S^3 \times S^3$ of the M-theory lift. The action of SO(4) is by left multiplication of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ on X and $\tilde{X}$. We can choose $\widetilde{\mathbb{Z}}_2$ to be $z_1 \to -z_1$ which corresponds to exchanging X and $\tilde{X}\beta$, where $\beta = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. The action of complex conjugation c_0 is given by

$$
X \mapsto X\beta\,, \qquad \tilde{X} \mapsto \tilde{X}\beta\,. \tag{3.13}
$$

Adding the M-theory circle adds to the asymptotic symmetry group a factor of $\mathrm{U}(1)^M$ which is given by right multiplication of X and $\tilde{X}$ by Θ as described above. Note that the orientifold action (3.13) inverts the M-theory circle as it should. Also note that the right action of Θ and β on $(X, \tilde{X})$ generates a larger group of symmetries which in particular contains the dihedral group D_N for any N. We denote the group generated in this way by G.

A further useful set of coordinates is given by writing $S^3 \times S^3$ as the quotient $\mathrm{SU}(2)^3/\,\mathrm{SU}(2)$,

$$
(g_1, g_2, g_3) \sim (g_1 g', g_2 g', g_3 g') \in \mathrm{SU}(2)^3/\,\mathrm{SU}(2)\,. \tag{3.14}
$$

The base of the conifold is obtained by reduction along the maximal torus of g_1. The explicit identification is

$$
\begin{aligned}
x &= g_2 g_3^{-1}\,, \\
\tilde{X} &= g_3 g_1^{-1}\,, \\
X &= x\tilde{X} = g_2 g_1^{-1}\,.
\end{aligned} \tag{3.15}
$$

Note that in these coordinates the flop is described by

$$
(g_1, g_2, g_3) \to (-\beta g_1, g_3, -g_2)\,. \tag{3.16}
$$

Topologically then, the boundary of our G_2-holonomy manifolds will be an orbifold of $S^3 \times S^3$. One interesting metric on $S^3 \times S^3$, which is the one underlying the G_2 metric on the spin bundle on S^3 [31, 17], has $\mathrm{SU}(2)^3 \times \Sigma_3$ symmetry, where $\mathrm{SU}(2)^3$ acts on the left in (3.14), and Σ_3 is the permutation of the three SU(2) factors [14]. Σ_3 is "spontaneously" broken in the interior, and only $\mathrm{SU}(2)^3 \times \mathbb{Z}_2$ are isometries of the full G_2 holonomy metrics. These metrics are relevant to the problem at infinite IIA coupling.

The metrics that are relevant for our discussion (at finite string coupling) have asymptotic symmetry group $(\mathrm{SU}(2) \times \mathrm{SU}(2) \ltimes \widetilde{\mathbb{Z}}_2 \times G)/_{\mathbb{Z}_2}$, where G was defined above as the group generated by $\mathrm{U}(1)^M$ and the orientifold action (3.13). The trivial $\mathbb{Z}_2$ is generated by the element $(-1,-1,-1) \in \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)^M$.

In the interior, various symmetry breaking patterns are possible. Moreover, in certain cases it happens that the symmetry group is enhanced to $\mathrm{SU}(2)^3$ in the deep interior.

3.3 Orientifolding

As in [12 – 14], we can consider dividing out by the action of a discrete group Γ preserving the G_2 metric. Of interest to us is the case that Γ is the (binary) dihedral group D_N. This group has generators a, and b satisfying the relations

$$a^{2N-4} = 1\,, \qquad b^2 = a^{N-2}\,, \qquad bab^{-1} = a^{-1}\,. \tag{3.17}$$

The group has a presentation

$$\mathbb{Z}_{2N-4} \longrightarrow D_N \longrightarrow \mathbb{Z}_2\,. \tag{3.18}$$

The dihedral group has a standard action on the three sphere coming from its embedding as a discrete subgroup of SU(2), namely

$$a = \begin{pmatrix} e^{\pi i/(N-2)} & 0 \\ 0 & e^{-\pi i/(N-2)} \end{pmatrix}, \qquad b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.19}$$

We will call the action of D_N on S^3 in which a and b are represented in this way as ρ. The quotient of our interest is obtained by letting D_N act as ρ on the right on both X and $\tilde{X}$, in the variables (3.15). The effect of the $\mathbb{Z}_{2N-4}$ factor in (3.18) is to reduce the length of the M-theory circle, thereby increasing the flux to $2N-4$ units. After reduction on $\mathrm{U}(1)^M$, the remaining $\mathbb{Z}_2$ sends $(x, y) \to (x, -y)$, so is indeed complex conjugation c_0. In terms of the variables $(g_1, g_2, g_3) \in \mathrm{SU}(2)^3/\,\mathrm{SU}(2)$, D_N acts as ρ on g_1 and trivially on g_2 and g_3.

We have discussed this action in detail in order to make the following point. There is another action, call it $\tilde{\rho}$, of D_N on S^3 in which b is represented by $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Acting with D_N as $\tilde{\rho}$ on the right of X and as ρ on the right of $\tilde{X}$ is equivalent to acting on g_1 as ρ and on g_2 via the central action, $a = 1$, $b = -1$. After reduction to $T^{1,1}$, the action is $(x, y) \to (-x, y)$, corresponding to c_π. We shall sometimes refer to this action as D'_N.

Note that the actions of D_N and D'_N can be conjugated into each other by a diffeomorphism of $S^3 \times S^3$, sending $(X, \tilde{X}) \to (X i\sigma_3, \tilde{X})$. However, this diffeomorphism is *not* an isometry of the boundary metric since $\mathrm{U}(1)^{\mathrm{phase}}$ is explicitly broken by the addition of flux.

We should now ask, what are the symmetries of the boundary metric which also preserve the orbifold group D_N or D'_N? In each case, $(\mathrm{SU}(2)\times\mathrm{SU}(2)\ltimes\widetilde{\mathbb{Z}}_2\times G)/_{\mathbb{Z}_2}$ is broken to $\mathrm{SO}(4)\ltimes\widetilde{\mathbb{Z}}_2\times\mathbb{Z}'_2$ where $\mathbb{Z}'_2$ is the centralizer of D_N or D'_N in G and is generated by

$$\sqrt{a}=\begin{pmatrix} e^{\pi i/2(N-2)} & 0 \\ 0 & e^{-\pi i/2(N-2)}\end{pmatrix}. \tag{3.20}$$

We can now make a few comments about the pattern of symmetry breaking by the various geometries in the interior. As we have already commented, we expect that $\widetilde{\mathbb{Z}}_2$ will be broken for the resolved conifold and unbroken for the deformed conifold. The two resolved conifolds are interchanged by the broken $\widetilde{\mathbb{Z}}_2$. On the other hand $\mathbb{Z}'_2$ is a subgroup of $\mathrm{U}(1)^M$ and will be unbroken whenever translation along the M-theory circle is a symmetry. This will fail to be true only for the deformed conifold with O6-plane on $S^3_>$ and $N=0$ or 1 D6-branes. Near the O6-plane we expect the geometry to look like Atiyah-Hitchin space or its double cover, wrapped on $S^3_>$. The M-theory circle is no longer an isometry direction, but for $N=1$, corresponding to the double cover, $\mathbb{Z}'_2$ is unbroken whilst for $N=0$, $\mathbb{Z}'_2$ is broken. For $N=0$, this leads to two distinct geometries interchanged by the broken symmetry generator. We shall return to this point and its interpretation in type IIA later.

4. M-theory geometry

In this section, we will flesh out our discussion of the M-theory geometries with some details of the explicit G_2 holonomy metrics. We shall describe in turn the three distinct classes of G_2 holonomy metrics relevant to our discussion. These are labeled $\mathbb{B}_7,\mathbb{A}_7$ and $\mathbb{D}_7$ and their various orbifolds correspond respectively to orientifolds of the deformed conifold with D6-branes, an O6-plane on $S^3_>$ with $N=0$ or 1 D6-branes and orientifolds of the resolved conifold with flux.

We present the details of these metrics in order to confirm the details of the symmetry breaking discussion of the previous section and also because the existence of the $\mathbb{A}_7$ metrics were not previously known in the literature. Furthermore, we show that there exists a normalizable harmonic two-form on $\mathbb{A}_7$ which leads to a massless U(1) gauge field in the IR. We can use this to rule out a smooth transition between these backgrounds and others with mass gap in the IR.

The reader who is not interested in the details of the metrics may wish to skip to the discussion of the topology of the various spaces in section 5.1.

4.1 Preliminaries

We recall the Euler angle representation of SU(2) matrices:

$$X=\begin{pmatrix} \cos\frac{\theta}{2}e^{\frac{i}{2}(\psi+\phi)} & -\sin\frac{\theta}{2}e^{-\frac{i}{2}(\psi-\phi)} \\ \sin\frac{\theta}{2}e^{\frac{i}{2}(\psi-\phi)} & \cos\frac{\theta}{2}e^{-\frac{i}{2}(\psi+\phi)}\end{pmatrix}, \tag{4.1}$$

where the coordinates take values

$$0\le\psi<4\pi\,,\qquad 0\le\phi<2\pi\,,\qquad 0\le\theta<\pi. \tag{4.2}$$

The associated Maurer-Cartan one-forms σ_a are defined by $X^{-1}dX = \frac{i}{2}\tau_a\sigma_a$ where τ_a are Pauli's matrices, and they satisfy $d\sigma_a = \frac{1}{2}\epsilon_{abc}\sigma_b\sigma_c$. They read

$$\sigma_1 = -\cos\psi\sin\theta d\phi + \sin\psi d\theta\,, \qquad \sigma_2 = -\sin\psi\sin\theta d\phi - \cos\psi d\theta\,, \qquad \sigma_3 = d\psi + \cos\theta d\phi. \tag{4.3}$$

One can write down the metric on $\mathbb{R}^4$ using polar coordinates r, θ, ϕ, ψ simply by relating the cartesian and polar coordinates as follows:

$$W = \begin{pmatrix} w_1 + iw_2 & -w_3 + iw_4 \\ w_3 + iw_4 & w_1 - iw_2 \end{pmatrix} = r \cdot X(\theta, \phi, \psi)\,, \qquad ds^2 = \frac{1}{2}\,\mathrm{Tr}(dW dW^\dagger) = dr^2 + \frac{r^2}{4}\sigma_a\sigma_a\,. \tag{4.4}$$

4.2 Deformed conifold with D6-branes and orientifold

Let us consider the M-theory lift of orientifolds of deformed conifold with D6-branes. The metrics should at least preserve the $\mathrm{SU}(2) \times \mathrm{SU}(2) \ltimes \widetilde{\mathbb{Z}}_2$ isometry of the deformed conifold, so we write the metrics in terms of SU(2) matrices $X, \tilde{X}$ and the associated Maurer-Cartan forms $\sigma_a, \tilde{\sigma}_a$. The following ansatz for a $\widetilde{\mathbb{Z}}_2$-symmetric metric was considered in [18]:

$$ds^2 = dr^2 + \sum_{i=1}^{3} A_i^2(\sigma_i - \tilde{\sigma}_i)^2 + \sum_{i=1}^{3} B_i^2(\sigma_i + \tilde{\sigma}_i)^2\,. \tag{4.5}$$

This metric is of G_2 holonomy provided the metric components obey

$$\begin{aligned}
\frac{dA_1}{dr} &= -\frac{1}{4}\left[\frac{A_1^2 - A_3^2 - B_2^2}{A_3 B_2} + \frac{A_1^2 - A_2^2 - B_3^2}{A_2 B_3}\right], \\
\frac{dA_2}{dr} &= -\frac{1}{4}\left[\frac{A_2^2 - A_1^2 - B_3^2}{A_1 B_3} + \frac{A_2^2 - A_3^2 - B_1^2}{A_3 B_1}\right], \\
\frac{dA_3}{dr} &= -\frac{1}{4}\left[\frac{A_3^2 - A_2^2 - B_1^2}{A_2 B_1} + \frac{A_3^2 - A_1^2 - B_2^2}{A_1 B_2}\right], \\
\frac{dB_1}{dr} &= \frac{1}{4}\left[\frac{A_2^2 + A_3^2 - B_1^2}{A_3 A_2} + \frac{B_1^2 - B_2^2 - B_3^2}{B_2 B_3}\right], \\
\frac{dB_2}{dr} &= \frac{1}{4}\left[\frac{A_3^2 + A_1^2 - B_2^2}{A_1 A_3} + \frac{B_2^2 - B_3^2 - B_1^2}{B_3 B_1}\right], \\
\frac{dB_3}{dr} &= \frac{1}{4}\left[\frac{A_1^2 + A_2^2 - B_3^2}{A_2 A_1} + \frac{B_3^2 - B_1^2 - B_2^2}{B_1 B_2}\right].
\end{aligned} \tag{4.6}$$

and the G_2 three-form is given by

$$\begin{aligned}
\Phi &= -e_1 e_2 e_3 + e_0 e_i e_{\tilde{i}} + \frac{1}{2}\epsilon_{ijk} e_i e_{\tilde{j}} e_{\tilde{k}} \\
&= -e_1 e_2 e_3 + e_0 e_1 e_{\tilde{1}} + e_0 e_2 e_{\tilde{2}} + e_0 e_3 e_{\tilde{3}} + e_1 e_{\tilde{2}} e_{\tilde{3}} + e_2 e_{\tilde{3}} e_{\tilde{1}} + e_3 e_{\tilde{1}} e_{\tilde{2}}\,.
\end{aligned} \tag{4.7}$$

The M-theory geometries $\mathbb{B}_7$ and $\mathbb{A}_7$ both take the above form. The metric for $\mathbb{B}_7$ has an additional $\mathrm{U}(1)^M$ symmetry corresponding to translation along M-theory circle, while $\mathbb{A}_7$ has no such U(1) symmetry.

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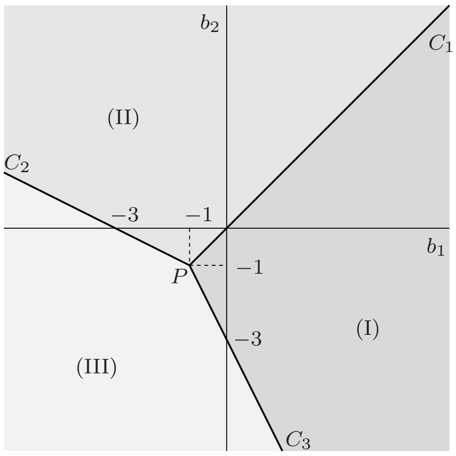

Figure 3: Parameter space of smooth initial data for $\mathbb{B}_7$. The metric has sensible asymptotics only when the initial data is on one of the three half lines $C_{1,2,3}$ bounding the regions (I), (II), (III).

Numerical analysis of the differential equations proceeds in the following way. We first perform a power series analysis at the origin $r = 0$ to find correct initial values for A_i, B_i that make the solution smooth there. We then let them evolve according to (4.6) and see if the metric asymptotes to that of a (conifold) $\times\, S^1$ with flux,

$$(A_1, A_2, A_3, B_1, B_2, B_3) \propto \frac{r}{6} \cdot (\sqrt{3}, \sqrt{3},\ 2, \sqrt{3}, \sqrt{3},\ 0)\,, \tag{4.8}$$

or that of a G_2 cone over $S^3 \times S^3$ (which describes the IIA theory at infinite string coupling),

$$(A_1, A_2, A_3, B_1, B_2, B_3) \propto \frac{r}{6} \cdot (\sqrt{3}, \sqrt{3}, \sqrt{3},\ 1,\ 1,\ 1)\,. \tag{4.9}$$

We will find that initial conditions which are regular at the origin do not always lead to a sensible asymptotic behavior.

4.2.1 Numerical analysis for $\mathbb{B}_7$

The correct initial data for smooth metric with an S^3 of unit size is given by the following power series

$$A_i = 1 + \frac{1}{16} r^2 + \mathcal{O}(r^3)\,, \qquad B_i = \frac{r}{4} + \frac{b_i}{192} r^3 + \mathcal{O}(r^4)\,, \qquad b_1 + b_2 + b_3 = -3\,. \tag{4.10}$$

This leading-order behavior uniquely determines the solution as a power series at $r = 0$.

The family of smooth initial data is parameterized by b_1, b_2. However, the metric asymptotes to (4.8) only when the initial data is on a half-line $C_1 : b_1 = b_2 \geq -1$ in the parameter space. (Owing to the permutation symmetry of $1, 2, 3$, there are three related families of initial conditions leading to metrics with sensible asymptotics. They are three half-lines C_1, C_2, C_3 meeting at $P : b_1 = b_2 = b_3 = -1$ as depicted in figure 3.)

Let us focus on the family of solutions C_1. Since we have fixed the size of the minimal S^3, the value of $b_1 = b_2$ determines the radius of the M-theory circle at infinity, which is roughly $\lim_{r\to\infty} B_3$. The radius blows up as $b_1 = b_2$ approach -1, and the solution has

SU(2) enhanced isometry and asymptotes to (4.9) at infinity. The solution is nothing but the familiar asymptotically conical G_2 holonomy metric on the spin bundle over S^3 [31, 17]. Alternatively, if we consider the family C_1 of rescaled solutions having a fixed radius of M-theory circle, then the limit of approaching P is the limit of vanishing S^3 at the center.

It is easy to see that, for the solutions on C_1, the equalities $A_1 = A_2, B_1 = B_2$ hold all the way along the radial evolution. So the manifold $\mathbb{B}_7$ has an additional $\mathrm{U}(1)^M$ isometry corresponding to translation along the M-theory circle. At the point P the solution satisfies $A_1 = A_2 = A_3, B_1 = B_2 = B_3$ along the radial evolution, so the $\mathrm{U}(1)^M$ is enhanced further to an SU(2).

The solutions at $r = 0$ take the form

$$\begin{aligned} ds^2 &\simeq dr^2 + \sum_{i=1}^{3}(\sigma_i - \tilde{\sigma}_i)^2 + \frac{r^2}{16}\sum_{i=1}^{3}(\sigma_i + \tilde{\sigma}_i)^2 \\ &= dr^2 + \sum_{i=1}^{3}\tilde{\Sigma}_i^2 + \frac{r^2}{16}\sum_{i=1}^{3}(2\Sigma_i - \tilde{\Sigma}_i)^2, \end{aligned} \tag{4.11}$$

where we introduced $Y = \tilde{X}X^{-1}$ and $\Sigma = XdX^{-1}, \tilde{\Sigma} = Y^{-1}dY$. Recalling the metric (4.4) on $\mathbb{R}^4$ one finds that the geometry is a smooth $\mathbb{R}^4$ bundle over S^3, where Y gives the base S^3 and (r, X) parameterize the fiber $\mathbb{R}^4$.

4.2.2 Orbifolding

The M-theory lifts of N D6-branes or (O6 + ND6) are orbifolds of $\mathbb{B}_7$ by A_{N-1} or D_N groups Γ. The orbifold group acts diagonally on X and $\tilde{X}$ from the right.

$$g \in \Gamma: \quad (X, \tilde{X}) \mapsto (Xg, \tilde{X}g)\,. \tag{4.12}$$

As we have seen, near the origin the coordinate $Y = \tilde{X}X^{-1}$ describes the base S^3 of finite volume, and X^{-1} describes the shrinking S^3. Γ acts on them as

$$g \in \Gamma: \quad (Y, X^{-1}) \mapsto (Y, g^{-1}X^{-1})\,. \tag{4.13}$$

Thus orbifolding gives a geometry which is $\mathbb{C}^2/\Gamma$ fibered over S^3.

The M-theory circle corresponds to the shift $\psi \to \psi + \alpha, \tilde{\psi} \to \tilde{\psi} + \alpha$. Its radius can be read off from the metric at infinity

$$\begin{aligned} ds^2 &= \cdots + A_3^2(\sigma_3 - \tilde{\sigma}_3)^2 + B_3^2(\sigma_3 + \tilde{\sigma}_3)^2, \\ &= \cdots + A_3^2(d\check{\psi} + \cos\theta d\phi - \cos\tilde{\theta}d\tilde{\phi})^2 + B_3^2(d\hat{\psi} + \cos\theta d\phi + \cos\tilde{\theta}d\tilde{\phi})^2\,. \end{aligned} \tag{4.14}$$

Here $\hat{\psi} = \psi + \tilde{\psi}$ is the coordinate on M-theory circle, and $\check{\psi} = \psi - \tilde{\psi}$ is one of the coordinates of $T^{1,1}$. From the periodicity of $\psi, \tilde{\psi}$ it follows that

$$0 \le \check{\psi} < 4\pi\,, \qquad 0 \le \hat{\psi} < 8\pi\,. \tag{4.15}$$

The RR charge is an integral of the field strength of the one-form potential A,

$$A = \frac{1}{4}(\cos\theta d\phi + \cos\tilde{\theta}d\tilde{\phi}) \tag{4.16}$$

over the S^2 defined by $\theta = \tilde{\theta}, \phi = \tilde{\phi}$ (One can check that its volume becomes zero at $r = 0$). We normalized A so that it appears in the M-theory metric as $(d\theta + A)^2$, where θ is of period 2π. The D6-brane charge is defined by

$$Q_{\rm D6} = \int_{S^2} \frac{dA}{2\pi}\,, \tag{4.17}$$

which is unity for $\mathbb{B}_7$ without orbifolding.

The A_{N-1} orbifold group is generated by $a = \exp(\frac{2\pi i \tau_3}{N})$. This simply shortens the period of M-theory circle to $1/N$, and therefore increases the D6-brane charge. For D_N orbifolds, Γ is generated by $a = \exp(\frac{i\pi\tau_3}{N-2})$ and $b = i\tau_2$. The element b acts as an antipodal map on S^2, and the D6-brane charge as counted in the covering space becomes

$$Q_{\rm D6} = 2 \times \int_{\mathbb{RP}^2} \frac{dA}{2\pi} = 2N - 4\,. \tag{4.18}$$

The action of b on the coordinates of the conifold was obtained in the previous section and is the complex conjugation c_0:

$$z_i \to \bar{z}_i\,. \tag{4.19}$$

The M-theory lift of the free orientifold of the deformed conifold with $2N-4$ D6-branes wrapping the $S^3_<$ uplifts to a different orbifold $\mathbb{B}_7/D'_N$. The generators a, b of D'_N act on $(X, \tilde{X})$ as

$$\begin{aligned} a:&\ \ (X,\tilde{X}) \mapsto \left(X \exp(\tfrac{i\pi\tau_3}{N-2}),\ \tilde{X}\exp(\tfrac{i\pi\tau_3}{N-2})\right), \\ b:&\ \ (X,\tilde{X}) \mapsto \left(X\cdot i\tau_2,\ \ -\tilde{X}\cdot i\tau_2\right). \end{aligned} \tag{4.20}$$

The generator b maps $z_i \mapsto -\bar{z}_i$, and in particular acts on the minimal $S^3_<$ as the antipodal map.

4.2.3 Numerical analysis for $\mathbb{A}_7$

Next we would like to describe the M-theory metrics corresponding to O6$^-$ and $N = 0$ or 1 D6-branes on $S^3_>$ of the deformed conifold. In this case the net flux from the O6$^-$ and D6 is negative and the M-theory geometry is expected to look locally like Atiyah-Hitchin space ($N = 0$) or its double cover ($N = 1$), wrapped on $S^3_>$.

We are looking for a G_2 metric which is asymptotic to (4.8) or (4.9). Furthermore, we expect the orbifold at infinity to be D'_4 for $N = 0$ and D'_3 for $N = 1$. Recall that D'_3 is generated by b which acts on $(X, \tilde{X})$ as in (4.20).

By analogy with the construction of Atiyah-Hitchin space [21], we expect that orbifolding by D'_3 is necessary in order to remove a conical singularity at the origin. This will be the case if $A_2 = r + \cdots$ near $r = 0$ so that the metric becomes

$$\begin{aligned} ds^2 =&\ A_1^2(\sigma_1 - \tilde{\sigma}_1)^2 + A_3^2(\sigma_3 - \tilde{\sigma}_3)^2 + B_1^2(\sigma_1 + \tilde{\sigma}_1)^2 + B_2^2(\sigma_2 + \tilde{\sigma}_2)^2 + B_3^2(\sigma_3 + \tilde{\sigma}_3)^2 + \\ &+ dr^2 + r^2(\sigma_2 - \tilde{\sigma}_2)^2 + \cdots\,. \end{aligned} \tag{4.21}$$

Here A_1, A_3, B_1, B_2, B_3 should be regarded as constants near $r = 0$. The first line gives the metric of a five-dimensional bolt which is topologically $S^2 \times S^3$. The second line yields a conical singularity which is removed by orbifolding by D'_3 as in (4.20). Note that in order

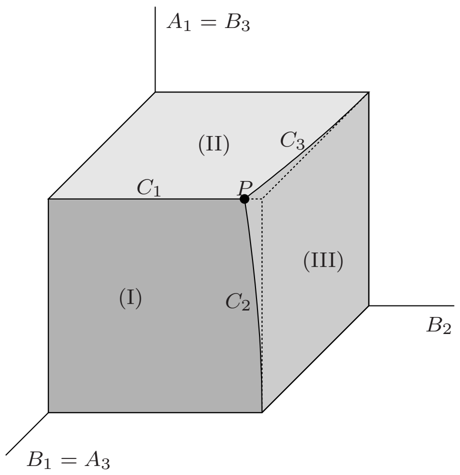

Figure 4: Parameter space of smooth initial data for the space $\mathbb{A}_7/\mathbb{Z}_2, \mathbb{A}_7$ with fixed scale. The metric behaves nicely when the initial data is chosen from one of the curves $C_{1,2,3}$ separating the three regions (I), (II), (III). The special point P corresponds to $(A_1, A_2, A_3, B_1, B_2, B_3) = (1, 0, 1, 1, .9171, 1)$.

for the shrinking S^1 to become an isometry direction at the origin (which is needed to avoid a singularity) we require $A_1 = B_3$ and $B_1 = A_3$ at $r = 0$.

The initial data for a smooth metric at $r = 0$ with five-dimensional bolt is

$$A_1 = B_3\,, \qquad B_1 = A_3\,, \qquad A_2 = 0\,. \tag{4.22}$$

Interestingly, if we set (A_3, B_1, B_2) all equal and much greater than $A_1(= B_3)$, then for small r, (A_3, B_1, B_2) stay almost constant while (A_1, A_2, B_3) approximately obey the equations for the Atiyah-Hitchin manifold [21]. However, the numerical analysis shows that such solutions do not extend toward large r and we should not try to impose these extra conditions.

The power series analysis shows that any initial values for (A_1, B_1, B_2) uniquely determine a solution, but this will have sensible asymptotics at large r only for some particular fine-tuned initial data. After fixing an overall scale, one gets a two-dimensional parameter space of initial conditions. We choose this to be three faces of a cube,

$$\{B_1 = 1,\ B_2, A_1 \in [0, 1]\}\,, \qquad \{B_2 = 1,\ B_1, A_1 \in [0, 1]\}\,, \qquad \{A_1 = 1,\ B_1, B_3 \in [0, 1]\}$$

as depicted in figure 4. One can numerically see that the solutions for generic initial conditions do not behave nicely at infinity: either B_1, B_2 or B_3 blows up much faster than the others. The generic initial conditions are grouped into regions (I), (II), (III) shown in figure 4 according to the asymptotics. The solution behaves nicely at large r if the initial condition is chosen on the curves $C_{1,2,3}$ separating three regions.

The curve C_1 is a straight line segment corresponding to the initial data

$$A_1 = B_3 = B_1 = A_3 = 1\,, \qquad 0 < B_2 \le 0.917\,, \qquad A_2 = 0\,.$$

Generic solutions with this initial condition asymptote locally to the conifold times an S^1 with flux,

$$(A_1, A_2, A_3, B_1, B_2, B_3) \propto (\sqrt{3}, 2, \sqrt{3}, \sqrt{3}, 0, \sqrt{3}) . \tag{4.23}$$

These solutions were discovered in [19] and named $\mathbb{C}_7$. They do not have the expected asymptotics (4.8) since the M-theory circle is in the B_2 direction rather than B_3. With these asymptotics, the orbifolding by D_3' acts as an orbifold on the base rather than as an orientifold. In addition, the solution has a U(1) isometry $A_1 = A_3, B_1 = B_3$ all along the flow, corresponding to translation along the M-theory circle. The solutions which we seek with O6$^-$-plane and $N = 0$ or 1 D6 are not expected to have such a $\mathrm{U}(1)^M$ isometry. As explained in [24] the solution $\mathbb{C}_7$ should be interpreted as the uplift of IIA on a manifold with local $\mathbb{P}^1 \times \mathbb{P}^1$ and unit flux through each $\mathbb{P}^1$. This solution will have no part to play in our current analysis.

The other two curves C_2, C_3 are related by the symmetry of the differential equation (4.6).

$$(A_1, A_2, A_3, B_1, B_2, B_3) \to (A_3, A_2, A_1, B_3, B_2, B_1) . \tag{4.24}$$

The solutions on C_2 flow to the correct asymptotic boundary conditions (4.8) and these turn out the solutions which we label $\mathbb{A}_7$ after an appropriate orbifolding. The solutions on C_3 describe the same set of solutions in different coordinates. We do not need to take account of them separately since the asymptotic boundary conditions (4.8) partially fix our choice of coordinates and the solutions on C_3 do not respect these asymptotics.

As we have discussed, it is necessary to orbifold the solutions by D_3' in order to remove a conical singularity at the origin. The resulting family of solutions $\mathbb{A}_7$ is asymptotically an orientifold of the conifold with RR charge (-1) and one can identify them with the M-theory lift of the O6$^-$+D6 system.

Further orbifolding by the right action of $(i\tau_3) \otimes (i\tau_3)$ on $(X, \tilde{X})$ halves the radius of the M-theory circle and thereby doubles the RR charge. Note that this does not lead to any orbifold singularities in the M-theory geometry $\mathbb{A}_7/\mathbb{Z}_2$. The enlarged orbifold group acts as D_4' on the boundary and this is the solution corresponding to an O6$^-$ and no D6 branes.

4.2.4 Normalizable harmonic two-form?

If there is such a two-form ω, it will lead to the presence of U(1) gauge dynamics in the IR. We expect that this should be the case for the M-theory lift of O6$^-$+D6 since this will have SO(2) = U(1) gauge group.

The two-form can be written locally as a derivative of a one-form. We assume that it takes the form

$$\omega = \sum_i d\left\{ f_i(r) \cdot (\sigma_i - \tilde{\sigma}_i) + g_i(r) \cdot (\sigma_i + \tilde{\sigma}_i) \right\} .$$

Instead of trying to solve $d\omega = d * \omega = 0$, we try to solve the simpler equation

$$*\omega = \alpha \Phi \wedge \omega , \tag{4.25}$$

where α is a real constant, requiring ω to fall into an irreducible representation of G_2. A little calculation shows that $\alpha = 1$ and f_i, g_i have to satisfy

$$\frac{1}{f_1}\frac{df_1}{dr} + \frac{1}{2}\left(\frac{A_1}{A_2 B_3} + \frac{A_1}{B_2 A_3}\right) = 0\,, \qquad \frac{1}{g_1}\frac{dg_1}{dr} + \frac{1}{2}\left(\frac{B_1}{A_2 A_3} - \frac{B_1}{B_2 B_3}\right) = 0\,, \qquad \text{etc.} \tag{4.26}$$

The equation shows the existence of six independent harmonic two-forms corresponding to f_i, g_i. The normalizability of each mode can be analyzed using the form of (A_i, B_i) at the origin and infinity.

For $\mathbb{A}_7$, the mode proportional to f_2 behaves at $r = 0$ and $r = \infty$ as

$$f_2 \overset{r\to 0}{\sim} 1 - \frac{1 + A_1^{-2}}{4} r^2 + \mathcal{O}(r^3)\,, \qquad f_2 \overset{r\to\infty}{\sim} \exp\left(-\frac{r}{2B_3}\right)\,, \tag{4.27}$$

and turns out to be normalizable. Note that this mode is projected out upon orbifolding further to get $\mathbb{A}_7/\mathbb{Z}_2$. For $\mathbb{B}_7$, none of these two-forms is normalizable. The absence of a normalizable harmonic two-form for $\mathbb{B}_7$ is puzzling, as discussed in [18], since one would expect the gauge group on N coincident D6-branes should be $\mathrm{U}(N)$ and its $\mathrm{U}(1)$ part should remain in the infrared limit.

4.3 Resolved conifold with flux and orientifold

For the M-theory lift of the resolved conifold with flux and orientifold, one expects the symmetry $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)^M$ but no $\widetilde{\mathbb{Z}}_2$ exchanging the two sets of Maurer-Cartan forms. The following ansatz for the metric was considered in [19]

$$ds^2 = dr^2 + a^2\{(\sigma_1 + g\tilde\sigma_1)^2 + (\sigma_2 + g\tilde\sigma_2)^2\} + b^2(\tilde\sigma_1^2 + \tilde\sigma_2^2) + c^2(\sigma_3 + g_3\tilde\sigma_3)^2 + f^2\tilde\sigma_3^2\,. \tag{4.28}$$

The metric is of G_2 holonomy provided $g = \frac{af}{2bc}, g_3 = -1 + 2g^2$ and

$$\begin{aligned} \dot a &= -\frac{c}{2a} + \frac{a^5 f^2}{8b^4c^3}\,, & \dot b &= -\frac{c}{2b} - \frac{a^2(a^2 - 3c^2)f^2}{8b^3c^3}\,, \\ \dot c &= -1 + \frac{c^2}{2a^2} + \frac{c^2}{2b^2} - \frac{3a^2 f^2}{8b^4}\,, & \dot f &= -\frac{a^4 f^3}{4b^4 c^3}\,. \end{aligned} \tag{4.29}$$

As initial conditions we put $a = c = 0,\ b = 1$ and $f = f_0$. The regularity at $r = 0$ requires

$$a = \frac{r}{2} + \mathcal{O}(r^2)\,, \qquad c = -\frac{r}{2} + \mathcal{O}(r^2). \tag{4.30}$$

If $f_0 < 1$, numerical solutions asymptote to (conifold) $\times S^1$,

$$(a, b, c) \sim r \cdot \left(\sqrt{\frac{1}{6}}, \sqrt{\frac{1}{6}}, -\sqrt{\frac{1}{9}}\right)\,, \qquad f \sim \text{const}\,, \qquad g \sim 0\,, \qquad g_3 \sim -1\,. \tag{4.31}$$

These solutions were named $\mathbb{D}_7$ in [19]. At $f_0 = 1$ the S^1 decompactifies and the solution coincides with the familiar asymptotically conical G_2 metric on the spin bundle over S^3.

Interestingly, the M-theory geometry has $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)^M$ symmetry and no extra $\mathrm{U}(1)^{\mathrm{phase}}$. It seems that the $\mathrm{U}(1)^{\mathrm{phase}}$ isometry corresponding to the phase rotation $z_i \to e^{i\alpha} z_i$ of the resolved conifold is broken in the presence of flux.

One can increase the flux or introduce an orientifold action simply by orbifolding the M-theory geometry. The orbifold group can be either A_{N-1} or D_N groups acting on $(X, \tilde{X})$ in the same way from the right, or it can be the group D'_N defined in (4.20). All these groups act freely. The RR charge is N for A_{N-1} orbifolds, $N-2$ for D_N orbifolds and $2-N$ for D'_N orbifolds.

5. Quantum moduli space

We are now in a position to study the quantum moduli space of supersymmetric orientifold vacua for different values of RR charge $Q = N - 2$. We first analyze them through the behavior of membrane instanton factors η_i on various classical M-theory geometries, as in [14]. We then study them using Vafa's exact superpotential.

From table 1, it appears that there are six classes of cases. $N - 2 > 1$ and $N < 0$ are "regular" and can be understood completely using M-theory arguments or the Vafa superpotential. $N = 3$ is somewhat special, but the same analysis applies. In all these cases, the moduli space turns out to be a copy of $\mathbb{P}^1$ with some number of marked points. When $N = 0, 1$ or 2, the moduli space of vacua consists of different branches and needs to be discussed separately.

5.1 M-theory lift

Here we try to study the quantum moduli space within the M-theory framework, through the behavior of various membrane instanton factors as holomorphic functions on moduli space. As preparation we need some basic results about the topology of the M-theory lifts.

We would like to find good chiral parameters to describe the classical and quantum moduli space. In an M-theory compactification on a G_2 manifold, X, the chiral parameters are of the form:

$$u = \exp\left(\int_Q (k\Phi_3 + iC)\right), \tag{5.1}$$

where the integral is taken over a non-trivial three-cycle $Q \in H_3(X, \mathbb{Z})$. Here, Φ_3 is the G_2 three-form and C is the M-theory three-form potential. k is a constant related to the membrane tension and u agrees with the exponential of the action of a membrane instanton wrapping an associative three-cycle homologous to Q.

Following [14] we introduce a set of chiral parameters corresponding to a basis of integral three-cycles in the boundary of the manifold X. The boundary of X is $Y_\Gamma \equiv (S^3 \times S^3)/\Gamma$ where Γ is the relevant orbifold group which is D_N (when $N > 2$) or $D'_{N'}$ (with $N' := 4 - N$ when $N < 2$). The space of integral three-cycles $H_3(Y, \mathbb{Z})$ is thus two-dimensional. On the various semi-classical branches, one three-cycle is 'filled in' so that $H_3(X, \mathbb{Z})$ is one-dimensional. Classically, the chiral parameter corresponding to the filled in cycle takes the value $\exp(0) = 1$. As explained in [14], this is subject to quantum corrections, but the classical analysis (supplemented with knowledge of the gauge theory) is reliable near limits of moduli space where the M-theory geometry is everywhere weakly curved (except for possible orbifold singularities). This gives information about the poles and zeros of the holomorphic parameters which is sufficient to reconstruct the exact moduli space.

Our first task, then, is to understand the third homology group of the boundary Y_Γ for $\Gamma = D_N$ or $D'_{N'}$ ($N' = 4 - N$). To describe $H_3(Y_\Gamma, \mathbb{Z})$, we follow [14]. Before modding out by Γ, the boundary is $Y = S^3 \times S^3$. This space can usefully be described in terms of the three SU(2) elements g_1, g_2, g_3 subject to the equivalence relation:

$$(g_1, g_2, g_3) = (g_1 h, g_2 h, g_3 h)\,. \tag{5.2}$$

Let $\hat{E}_i \subset \mathrm{SU}(2)^3$ be the i^{th} copy of SU(2) — so $\hat{E}_1$ is the set $(g, 1, 1)$, $g \in \mathrm{SU}(2)$. In Y, the $\hat{E}_i$ project to cycles E_i obeying

$$E_1 + E_2 + E_3 = 0\,. \tag{5.3}$$

Under the orbifold projection $Y \to Y_\Gamma$ for $\Gamma = D_N$, the E_i are mapped to cycles in Y_Γ which we label E'_i. The map of E_1 to E'_1 is the $(4N-8)$-fold cover, $S^3 \to S^3/D_N$, whilst E_2 and E_3 are mapped diffeomorphically to E'_2 and E'_3. Thus we have

$$(4N-8)E'_1 + E'_2 + E'_3 = 0\,. \tag{5.4}$$

Y_Γ is simply the product $E'_1 \times E'_2$ and these cycles generate $H_3(Y_\Gamma, \mathbb{Z})$.

For $\Gamma = D'_{N'}$, E_2 and E_3 are again mapped diffeomorphically to E'_2 and E'_3 in Y_Γ. To find another cycle, instead of E_1 we consider a little different three sphere in Y;

$$\left(g, g^{-1} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} g, 1 \right), \qquad g \in \mathrm{SU}(2)\,. \tag{5.5}$$

Left multiplication by $D_{N'}$ elements on g corresponds to the $D'_{N'}$ action on Y. Thus, this S^3 defines a $(4N'-8)$-fold cover over a cycle $E \cong S^3/D_{N'}$ in Y_Γ. Note that the cover S^3 has the same homology class as E_1. This is because the inverse $g \mapsto g^{-1}$ reverses the orientation of SU(2) and therefore the g_2 part in (5.5) does not contribute in homology. Using (5.3), we find the homology relation in Y_Γ,

$$(4N'-8)E + E'_2 + E'_3 = 0\,.$$

Denoting the orientation reversal of E by E'_1, we find the same relation as (5.4) with $N = 4 - N'$. Y_Γ is just the product $E'_1 \times E'_2$ and these cycles generate $H_3(Y_\Gamma, \mathbb{Z})$.

Next we would like to describe the behavior of the holomorphic parameters

$$\eta_i = \exp\left(\int_{E'_i} (k\Phi_3 + iC) \right) \tag{5.6}$$

at the various semi-classical limits of moduli space. The homology relation (5.4) implies the following relation of η_i's

$$\eta_1^{4N-8}\eta_2\eta_3 = 1\,. \tag{5.7}$$

As remarked in [14], one must be careful about the definition of $\exp(i\int_{E'_i} C)$ — this must include the sign factor of the fermionic determinant, and there is a potential sign error in the right hand side of (5.7). It is shown in [14] that the error is absent for $\Gamma = D_N$, and the same proof applies also to $\Gamma = D'_{N'}$. The reason is that, just as in D_N case, Y_Γ is the product of spin manifolds $Y_1 = S^3/D_{N'}$ and $Y_2 = S^3$, and that E'_i are all transverse to a Spin(3) sub-bundle of $TY_\Gamma \cong TY_1 \oplus TY_2$ — a slight rotation of TY_1 in the direction of TY_2.

Thus, the relation (5.7) exactly holds at each of the semi-classical points, and therefore by holomorphy, everywhere on the moduli space. We shall determine the orders of the poles and zeroes of the η_i as functions on moduli space, following the argument in [14]. Part of this information is obtained classically by counting the number of times which a boundary cycle E'_i wraps the minimal three-cycle of the geometry. However, if the geometry has an orbifold singularity, the classical analysis is modified by strong coupling effects at low energies.

Near the deformed classical points corresponding to orbifolds of $\mathbb{B}_7$ with large S^3, E'_1 is filled in whilst E'_2 and E'_3 grow large with opposite orientation. Thus

$$\begin{array}{lllll}
(N>2) & \eta_1 \to 1\,, & \eta_2 \to 0^h\,, & \eta_3 \to \infty^h\,, & h \equiv \check{h}_{\mathrm{SO}(2N)} = 2N-2\,,\\
 & \eta_1 \to -1\,, & \eta_2 \to 0^{2h'}\,, & \eta_3 \to \infty^{2h'}\,, & h \equiv \check{h}_{\mathrm{Sp}(N-4)} = N-3\,,\\
(N<2) & \eta_1 \to 1\,, & \eta_2 \to \infty^{\tilde{h}}\,, & \eta_3 \to 0^{\tilde{h}}\,, & \tilde{h} \equiv \check{h}_{\mathrm{SO}(4-2N)} = 2-2N\,,\\
 & \eta_1 \to -1\,, & \eta_2 \to \infty^{2\tilde{h}'}\,, & \eta_3 \to 0^{2\tilde{h}'}\,, & \tilde{h}' \equiv \check{h}_{\mathrm{Sp}(2-N)} = 3-N\,.
\end{array} \tag{5.8}$$

Note that, as explained in [14], the $\pm$ sign for η_1 corresponds to a choice of discrete flux which leads to SO or Sp gauge theory, and the order of zeroes or poles is related to the degeneracy of vacua of the corresponding super Yang-Mills theory.[3] Also, we have chosen the orientations of cycles E'_i so that the family of vacua preserves the same supersymmetry for all N, in accordance with table 1 in the introduction. One can see that the orders of zeroes or poles depend linearly on N under this choice.

On the two resolved vacua described by orbifolds of $\mathbb{D}_7$ we have either E'_2 or E'_3 filled in. In the former case the minimal cycle is homologous to $E'_1 = -\frac{1}{4N-8}E'_3$ and

$$\eta_1 \to \infty\,, \qquad \eta_2 \to 1\,, \qquad \eta_3 \to 0^{4N-8}\,, \tag{5.9}$$

whilst in the latter case one has

$$\eta_1 \to 0\,, \qquad \eta_2 \to \infty^{4N-8}\,, \qquad \eta_3 \to 1\,. \tag{5.10}$$

The pole turn into zero and vice versa as $N-2$ flips sign, due to our choice of the orientations of E'_i.

On the classical vacua corresponding to $\mathbb{A}_7/\mathbb{Z}_2$ and its double cover $\mathbb{A}_7$, we expect a similar behavior to the deformed classical vacua except that the large S^3 should have opposite orientation. In both cases, E'_2 and E'_3 wrap the minimal cycle once and so the relevant poles and zeroes are simple.

$$\eta_1 \to 1\,, \qquad \eta_2 \to 0\,, \qquad \eta_3 \to \infty\,. \tag{5.11}$$

[3]The extra factor of 2 for the symplectic group was explained in [14] for the $N > 2$ cases with O6-plane at S^3, using an argument in which D6-branes are deformed away from the orientifold plane. In fact, we need this extra factor also for the $N < 2$ cases with free orientifold with minimal $\mathbb{RP}^3$. Derivation that applies to both cases is given in section 5.2.

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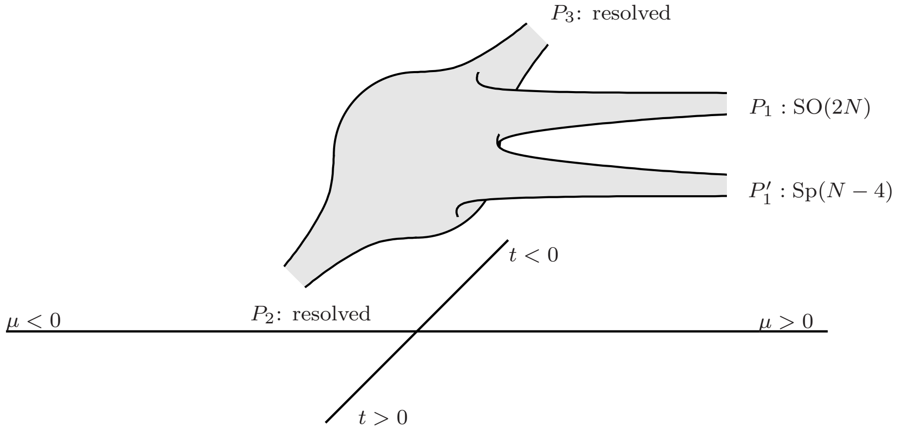

Figure 5: The quantum moduli space for the cases $N>3$. Here μ and t are the sizes of deformation and resolution.

Now, let us glue together the local behaviors of η_i to get the curve describing the moduli space.

5.1.1 $N>3$

The quantum moduli space of the orientifold plane system has been analyzed in [14] for the cases with $N>3$ D6-branes. It consists of four classical geometries connected in the quantum moduli space (see figure 5) at the points $\eta_1=\pm 1, 0, \infty$. The point $\eta_1=1$ corresponds to an O6$^-$-plane and N D6-branes wrapping the $S^3_>$ of the deformed conifold. The gauge group at this point is SO$(2N)$. The point $\eta_1=-1$ corresponds to an O6$^+$-plane and $N-4$ D6-branes wrapping the $S^3_>$ of the deformed conifold that has a gauge group Sp$(N-4)$. The points $\eta_1=0,\infty$ correspond to the two resolved conifolds with $N-2$ units of RR flux.

The quantum curve can be obtained by looking at the poles and zeros of the η_i functions, as explained above:

	P_1	P_1'	P_2	P_3
η_1	1	-1	0	∞
η_2	0^{2N-2}	$0^{2(N-3)}$	∞^{4N-8}	1
η_3	∞^{2N-2}	$\infty^{2(N-3)}$	1	0^{4N-8}

From this table it is easy to deduce the quantum curve:

$$\begin{aligned}\eta_2 &= \eta_1^{-(4N-8)}(\eta_1-1)^{(2N-2)}(\eta_1+1)^{(2N-6)}\,,\\ \eta_3 &= (\eta_1-1)^{-(2N-2)}(\eta_1+1)^{-(2N-6)}.\end{aligned} \tag{5.12}$$

Notice that for this charge there is no classical configuration involving anti-D6-branes wrapped on $S^3_<$. They would preserve the same supersymmetry but the charge would be negative.

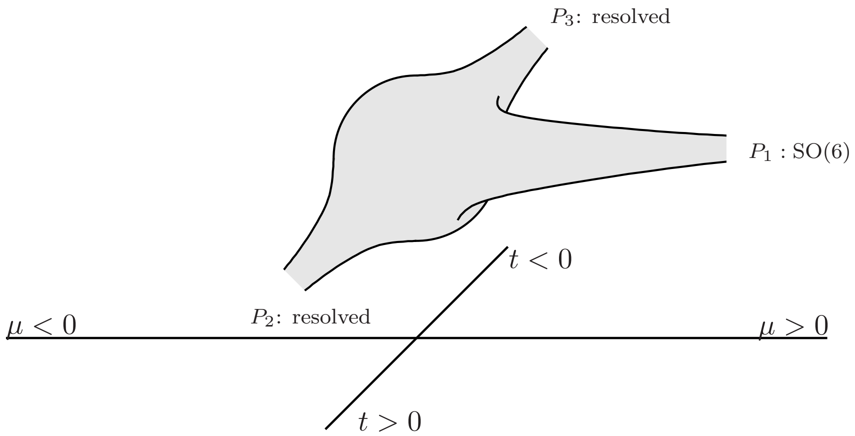

Figure 6: The quantum moduli space for the case $N = 3$.

5.1.2 $N = 3$

The case $N = 3$ with orientifold planes coincides with the A_3 case (see figure 6). The quantum moduli space has three points with classical descriptions at $\eta_1 = 1, 0, \infty$. The point $\eta_1 = 1$ corresponds to an O6$^-$-plane and three D6-branes wrapping the $S^3_>$ of the deformed conifold. The gauge group at this point is SO(6) $\sim$ SU(4), that has $h = 4$ vacua. The points $\eta_1 = 0, \infty$ correspond to the two resolved conifolds with 1 unit of RR flux. As expected the quantum curve is the same as in the A_3 case:

$$\begin{aligned}\eta_2 =& \eta_1^{-4}(\eta_1 - 1)^4\,, \\ \eta_3 =& (1-\eta_1)^{-4}\,.\end{aligned} \tag{5.13}$$

The poles and the zeros for the $N = 3$ case can be summarized in the following table:

	P_1	P_2	P_3
η_1	1	0	∞
η_2	0^4	∞^4	1
η_3	∞^4	1	0^4

As in the previous case there are no classical configurations involving anti-D6-branes and preserving the same supersymmetries.

5.1.3 $N = 2$

This case is truly exceptional, because the boundary of the relevant M-theory geometry is not an orbifold of $S^3 \times S^3$, but rather

$$\left(\frac{S^2 \times S^1}{\mathbb{Z}_2}\right) \times S^3\,. \tag{5.14}$$

We denote the first factor by E'_1 and the second by $E'_2 = -E'_3$, and define the associated membrane instanton factors η_i as explained before.

The classical point P_1 corresponding to an O6$^-$-plane and two D6-branes wrapped on $S^3_>$ supports an SO(4) gauge theory. Although $E_2' = -E_3'$ wraps the minimal cycle once, the corresponding instanton factors η_2, η_3 develop double zero and pole there due to the SO(4) gauge dynamics. Here we determined the degeneracy of vacua from $\check{h}_{\mathrm{SO}(N)} = 2N - 2 = 2$ for $N = 2$, though SO(4) super Yang-Mills theory actually has four degenerate vacua. At the two resolved classical points $P_{2,3}$ the η_1 has a simple zero or pole, whereas $\eta_{2,3}$ remain finite.

In addition, we have geometries with large $\mathbb{RP}^3 \equiv S^3_</\mathbb{Z}_2$ corresponding to free orientifold of deformed conifold. We claim that there are two distinct classical points with large $\mathbb{RP}^3$. The two will differ, from type-IIA viewpoint, in the action on the Chan-Paton indices when one wraps some D6-branes on $\mathbb{RP}^3$. They should also be distinguished by the discrete torsion of NSNS B-field or M-theory three-form potential. Indeed, the third homology group $H_3(X, \mathbb{Z})$ of the relevant M-theory geometry

$$X : \ \ \mathbb{R}^3 \times S^1 \ \rightarrow \ \mathbb{RP}^3$$

is $\mathbb{Z} \oplus \mathbb{Z}_2$ and therefore has a torsion part, since the third homology group of the boundary (5.14) is $\mathbb{Z} \oplus \mathbb{Z}$ and only *twice* the first generator is trivial in X. Therefore, one has classically the choice[4]

$$\int_{E_1'} C = 0 \ \text{ or } \ \pi \quad (\mathrm{mod}\ 2\pi)\,. \tag{5.15}$$

We denote the corresponding two classical points by P_1' and P_1''.

The table of singularities reads:

	P_1	(P_1)	P_1'	P_1''	P_2	P_3
η_1	1	1	1	-1	0	∞
η_2	0^2	0^2	∞^2	∞^2	1	1
η_3	∞^2	∞^2	0^2	0^2	1	1

Here we included in the second column the contribution of the two SO(4) vacua that we missed by the counting based on $\check{h}_{\mathrm{SO}(4)} = 2$.

From exact superpotential explained later, one can derive the quantum curve:

$$\begin{aligned} \eta_2 &= (\eta_1 - 1)^2 (\eta_1 + 1)^{-2}\,, \\ \eta_3 &= (\eta_1 - 1)^{-2} (\eta_1 + 1)^2\,. \end{aligned} \tag{5.16}$$

This accounts for only four classical points P_1, P_1'', P_2, P_3. It is thus expected that the moduli space consists of two branches, one of which is the curve given above whereas the other contains the two missing SO(4) vacua as well as the classical point P_1'. The latter branch is most likely a cylinder $\mathbb{C}^\times$. If we parameterize it by z the η_i's on this branch will be given by

$$\eta_1 = 1\,, \qquad \eta_2 = z^2\,, \qquad \eta_3 = z^{-2}\,. \tag{5.17}$$

The two branches meet at the classical point with an O6$^-$-plane and two D6-branes on S^3. The structure of quantum moduli space is summarized in figure 7.

[4] Alternatively, we may look at the twisted second homology of the IIA reduction of $\tilde{X}$. We again find the torsion $H_2(\tilde{X}, \tilde{\mathbb{Z}}) = \mathbb{Z}_2$ as $\tilde{X}$ contains an $\mathbb{RP}^3$.

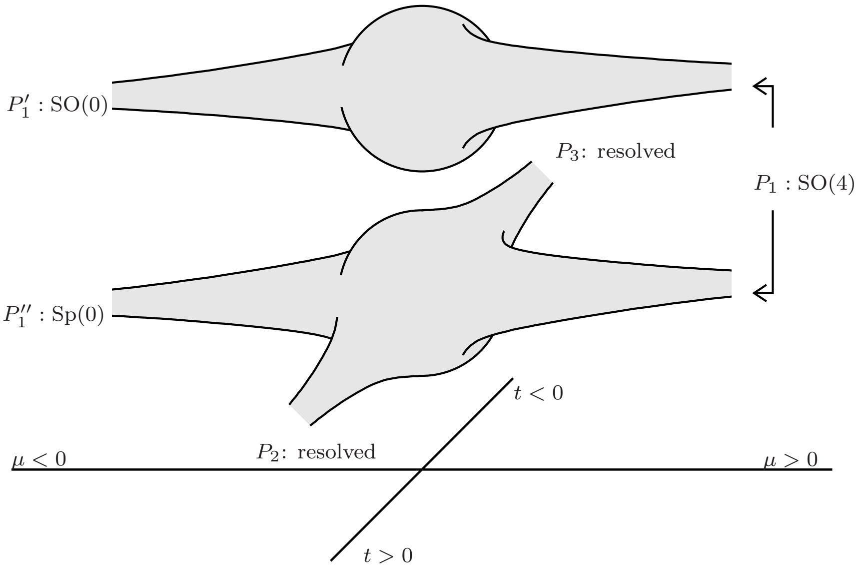

Figure 7: The quantum moduli space for the case $N = 2$.

5.1.4 $N = 1$

There are five classical points: the one corresponding to an O6$^-$-plane and a D6-brane wrapping the $S^3_>$ which we denote by P_1, the free orientifold plus two D6-branes wrapping the $S^3_<$ which are denoted by P_1' or P_1'' according to the gauge group being SO(2) or Sp(1), and the two resolved vacua (P_2, P_3). The orders of zeroes or poles of η_i is summarized as follows:

	P_1	P_1'	P_1''	P_2	P_3
η_1	1	1	-1	0	∞
η_2	0	∞	∞^4	0^4	1
η_3	∞	0	0^4	1	∞^4

The first two have massless U(1) gauge bosons at low energies whereas the other three have a mass gap. The moduli space should therefore consist of two smooth components, one for vacua with U(1) and the other for mass-gapped vacua. From the analysis of zeroes and poles we expect that the branch of vacua with U(1) is a cylinder $\eta_2\eta_3 = 1$, and the mass-gapped branch is given by the curve

$$\begin{aligned} \eta_2 &= \eta_1^4(\eta_1+1)^{-4}\,, \\ \eta_3 &= (\eta_1+1)^4\,. \end{aligned} \tag{5.18}$$

We propose that the two branches meet at a phase transition point, as shown in figure 8. As will be discussed in detail in section 6, one can get the exact branch structure by following a long chain of dualities to go to a “mirror” IIB theory. Let us summarize here the main points.

The idea is to consider another type-IIA limit by reduction on a different circle and then take its type-IIB mirror. This is the route found in [32] and the technique is further

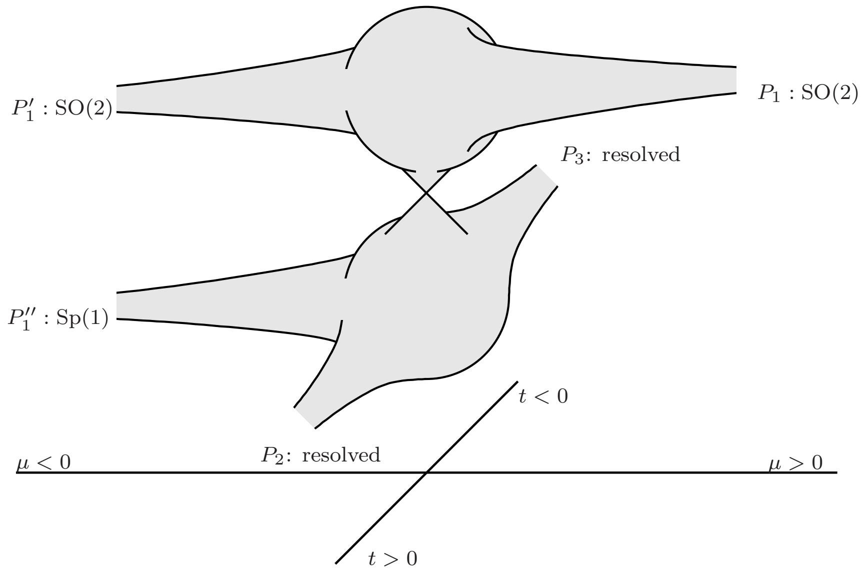

Figure 8: The quantum moduli space for the case $N = 1$. The two branches are connected at a phase transition point, as we will see in the next section.

developed in [24] on which the present computation is based. The type-IIB dual is the non-compact Calabi-Yau ($\xi, \eta \in \mathbb{C}$; $x, y \in \mathbb{C}^\times$)

$$\xi\eta = F(x, y) := y^2 - 2sxy + x^3 - 2x^2 + x\,,$$

parameterized by s^2, together with a D5-brane located at a line $\eta = 0$, ξ free. The D5-brane position is parameterized by a point $(x, y) = (x_\mathbf{D}, y_\mathbf{D})$ of the Riemann surface $F(x, y) = 0$, which is generically genus one and has three punctures **A**, **B**, **C** at $(x, y) = (1, 0), (0, 0), (\infty, \infty)$. The modulus s is a normalizable dynamical variable but $\mathbf{D} = (x_\mathbf{D}, y_\mathbf{D})$ is a coupling constant. The presence of the D5-brane generates a superpotential $W(x_\mathbf{D}, s)$, and the extremization $\partial_s W = 0$ relates **D** and s as follows: When the curve $F(x, y) = 0$ has genus one, the D5-position **D** is determined by s^2. This is what we call the $g = 1$ branch. At $s^2 = -4$, the curve degenerates to genus zero and then **D** is free to move on this curve. We call this the $g = 0$ branch.

On the $g = 1$ branch there is a U(1) vector multiplet which is an $\mathcal{N} = 2$ superpartner of the complex structure modulus s. This corresponds to the upper branch in figure 8. The points P_1 (O6$^-$ with $D6$) and P_1' (free orientifold with SO(2) D6's on $\mathbb{RP}^3$) correspond respectively to the large complex structure limit $s^2 = \infty$ and the "orbifold limit" $s^2 = 0$. The $g = 0$ branch has no massless vector and corresponds to the lower branch in figure 8. The point P_1'' (free orientifold with Sp(1) D6's on $\mathbb{RP}^3$) corresponds to the limit where **D** approaches the marked point **A** while the points P_2 and P_3 (two resolved conifolds) correspond to $\mathbf{D} \longrightarrow \mathbf{B}, \mathbf{C}$. This branch structure is reminiscent of the result of [33], where the behavior of vacua of $\mathcal{N} = 1$ gauge theory with an adjoint matter was studied on the space of superpotential couplings.

Let us now focus on the transition point where the two branches meet. From the $g = 1$ side, this is the point $s^2 = -4$ where a linear combination of the A cycle and B cycle of the torus degenerates, and type-IIB D3-brane wrapped on this vanishing cycle becomes massless. Such D3-brane states, which constitute a charged hypermultiplet $(M, \tilde{M})$, must be included in the low energy effective theory near the point $s^2 = -4$. The effective superpotential is then given by

$$W_{\text{eff}} = W(x_{\mathbf{D}}, s) + (s^2 + 4)\tilde{M}M \,. \tag{5.19}$$

Variation with respect to the normalizable variables s, M and $\tilde{M}$ yields

$$\partial_s W(x_{\mathbf{D}}, s) + 2s\tilde{M}M = 0\,, \qquad (s^2+4)\tilde{M} = 0\,, \qquad (s^2+4)M = 0\,.$$

The solutions with $M = \tilde{M} = 0$ leave the U(1) gauge symmetry unbroken and lie in the $g = 1$ branch. There are other solutions with $s^2 = -4$ in which $\tilde{M}M$ is determined by $x_{\mathbf{D}}$ through the first equation and its non-zero value higgses the U(1). They constitute the $g = 0$ branch.

In the original type-IIA or M-theory description, what are the particles that become massless at the transition point? type-IIB D3-branes wrapped on A and B cycles of the curve (plus two other directions) near the large complex structure limit $s^2 = \infty$ correspond in M-theory on $\mathbb{A}_7$ to a membrane wrapped on the S^2 bolt of a Dancer's fiber and a fivebrane wrapped on the $T^{1,1}$ bolt of $\mathbb{A}_7$. The corresponding objects in type-IIA orientifold with O6$^-$+D6 wrapped on S^3 (point P_1) are essentially the non-BPS states discussed in [34]: a membrane wrapped on the non-holomorphic S^2 bolt of Dancer's manifold is a massive oscillation mode of the open string stretched between a D6 and its orientifold image, while a fivebrane wrapped on the bolt is a non-BPS threebrane whose tension is $1/\ell_s^4$ in the strong coupling limit. Thus, the charged particle responsible for the transition to the confining branch is the electrically charged massive open string mode on D6 or the magnetic non-BPS threebrane wrapped on S^3, or some dyonic bound state. Which one becomes massless at $s^2 = -4$, is an interesting question although somewhat ambiguous because of monodromies around the other special points in moduli space.

5.1.5 $N = 0$

There are at least five classical points: an O6$^-$-plane on $S^3_>$ (P_1), the free orientifold plus two D6-branes on $S^3_<$ (P_1' or P_1'' depending on the gauge groups SO(4) or Sp(2)), and the two of resolved vacua $P_{2,3}$. The orders of zeroes and poles read

	P_1	P_1'	(P_1')	P_1''	P_2	P_3
η_1	1	1	1	-1	0	∞
η_2	0	∞^2	∞^2	∞^6	0^8	1
η_3	∞	0^2	0^2	0^6	1	∞^8

Here we included the contribution of two SO(4) vacua at P_1' that are missed by the counting based on $\check{h}_{\mathrm{SO}(4)} = 2$. It follows that the four points P_1', P_1'', P_2, P_3 can live on a single Riemann sphere as described by the equations

$$\begin{aligned} \eta_2 &= \eta_1^8(\eta_1 - 1)^{-2}(\eta_1+1)^{-6}\,, \\ \eta_3 &= (\eta_1 - 1)^2(\eta_1+1)^6\,. \end{aligned} \tag{5.20}$$

This curve actually follows also from Vafa's exact superpotential. A natural guess then is that there is another branch of moduli space containing all these missing vacua.

As a non-trivial check for this guess, let us consider the orders of zeroes or poles of the functions η_i on the new branch. At the SO(4) classical point on the new branch one should have $\eta_2 \sim 0^2$, $\eta_3 \sim \infty^2$ to account for the missing vacua (third column of the table above). On the other hand, at the vacuum P_1 corresponds to an M-theory geometry $\mathbb{A}_7/\mathbb{Z}_2$ which has a finite $S^3 \simeq E_2' \simeq -E_3'$, so that $\eta_2 \sim \infty$, $\eta_3 \sim 0$ at the corresponding classical point. Therefore, the numbers of poles and zeroes agree on the "new" branch if there are two distinct classical points corresponding to an O6$^-$-plane on $S^3_>$. Remarkably, this is in agreement with the fact that the corresponding M-theory geometry $\mathbb{A}_7/\mathbb{Z}_2$ spontaneously breaks the $\mathbb{Z}_2'$ symmetry of section 3.3.

As we have seen in section 3.3, the asymptotic symmetry of M-theory geometry is $\mathrm{SO}(4) \ltimes \widetilde{\mathbb{Z}}_2 \times \mathbb{Z}_2'$. In the interior of some solutions, a part of $\tilde{\mathbb{Z}}_2 \times Z_2'$ is broken. The two resolved vacua are permuted under $\tilde{\mathbb{Z}}_2$ defined in (3.16), while all the deformed vacua are invariant. On the other hand, $\mathbb{Z}_2'$ is the centralizer of the D_4' orbifold group and is identified with a half-period shift along the M-theory circle. As such, it is broken in the solution $\mathbb{A}_7/\mathbb{Z}_2$ corresponding to an O6$^-$-plane on $S^3_>$ while all other classical solutions are invariant. Thus, there should be two classical points corresponding to an O6$^-$-plane on $S^3_>$, as claimed.

Are the two branches connected? From what we discussed above it is clear that $\widetilde{\mathbb{Z}}_2$ acts non-trivially on the branch containing resolved vacua, while $\mathbb{Z}_2'$ acts non-trivially on the other branch. Conversely, $\mathbb{Z}_2'$ should act trivially on the branch containing resolved vacua and $\widetilde{\mathbb{Z}}_2$ should act trivially on the other branch, under the assumption that both branches are $\mathbb{P}^1$'s with various marked points, (since both $\mathbb{Z}_2$'s act holomorphically on moduli space and fix at least three points on the appropriate branches.) The two branches can therefore be connected only through points invariant under both $\mathbb{Z}_2$'s. Since the classical points $\eta_1 = \pm 1$ are fixed by $\widetilde{\mathbb{Z}}_2$ and they are the only fixed points on the resolved branch, one can conclude immediately that the two branches are not connected through interior points. The moduli space is thus made of two disjoint branches as depicted in figure 9.

One might have guessed that a phase transition as in the $N = 1$ case would connect the two branches because the classical moduli space is connected. Such a phase transition would be characterized by the emergence of massless particles. For the $N = 1$ case, the relevant particle in the M-theory framework is either a five-brane wrapped on the $T^{1,1}$ bolt of $\mathbb{A}_7$ or a membrane wrapped on the S^2. However, the corresponding cycles for the case $N = 0$ are both non-orientable, so there will be no massless particles when they shrink to zero size.

$\mathbb{Z}_2'$ in type IIA.[5] The presence of the discrete symmetry in $\mathbb{Z}_2'$ might at first sight appear puzzling from the type-IIA perspective: A half-period shift of the M-theory circle descends in type IIA to D0-brane charge modulo 2, which according to our discussion should be preserved in front of $N > 0$ D6-branes wrapped on top of the O6$^-$, and broken precisely for $N = 0$. Naively, this appears to be in conflict with the familiar classification of D-brane charge in string theory. A perturbative analysis in flat space shows that the tachyonic ground state of the 0–0 strings is in the symmetric representation. The tachyon

[5] Discussions with S. Hellerman were instrumental in shaping the following arguments.

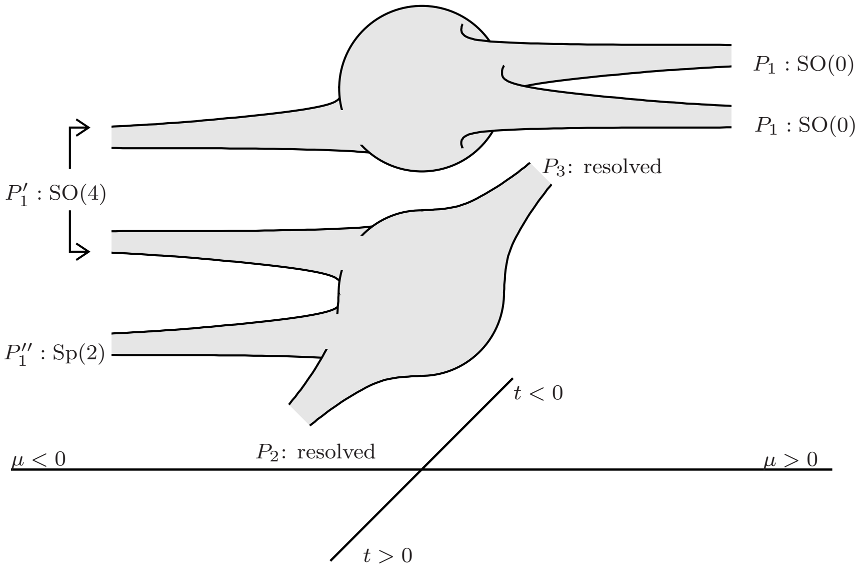

Figure 9: The quantum moduli space for the case $N = 0$.

is therefore not orientifolded out even for a single D0-brane, which should therefore not produce a conserved charge. This situation is T-dual to the D3-brane in type I, and in distinction to the D(-1)-brane there, for which the ground state is in the anti-symmetric representation.

Wrapping on S^3 does not eliminate the tachyon, and one would therefore not expect a stable D0-brane in our backgrounds, for any N. On the other hand, the M-theory analysis solidly establishes the existence of $\mathbb{Z}_2'$ for $N > 0$, and its breaking for $N = 0$.

To reconcile the two points of view, we note that the analog of $\mathbb{Z}_2'$ can already be seen in the context of O6$^-$/D6 systems in flat space and their M-theory lifts, with the same breaking pattern. The discrete symmetry in this case can be nicely interpreted from the perspective of D2-brane probes [35, 36]. (Its presence was also noticed, for example, in the D0-brane scattering analysis of [37].) The worldvolume theory of a D2-brane pair is a $3d$ $\mathcal{N} = 4$ supersymmetric Sp(1) gauge theory with $2N$ half-hypermultiplets in the fundamental representation. The hypers are from the 2-6 strings and their masses parameterize the position of the D6-branes. Now, one may interpret the $\mathbb{Z}_2'$ as the global symmetry extending SO$(2N)$ by O$(2N)$. It acts as the sign flip of just one of the $2N$ half-hypermultiplets. That this correspond to a half-period shift of the M-theory circle follows from the fact that, in an odd monopole background, the fermion zero mode measure is odd under this symmetry. When some of the D6-brane pairs are on top of the O6-plane, the sign flip of one of the corresponding massless half-hypermultiplets is a symmetry of the system. When all D6-branes are away from the O6, all the masses are turned on, and the symmetry is broken. Namely, $\mathbb{Z}_2'$ is a symmetry only when at least a single D6-brane is *exactly on top* of the O6-plane, and broken otherwise. This is indeed the breaking pattern we have found in the curved background.

Returning to the interpretation of $\mathbb{Z}_2'$ as "D0-brane number modulo 2", we note that this is a good quantum number far away from the O6-plane. Indeed, a D0-brane is then far

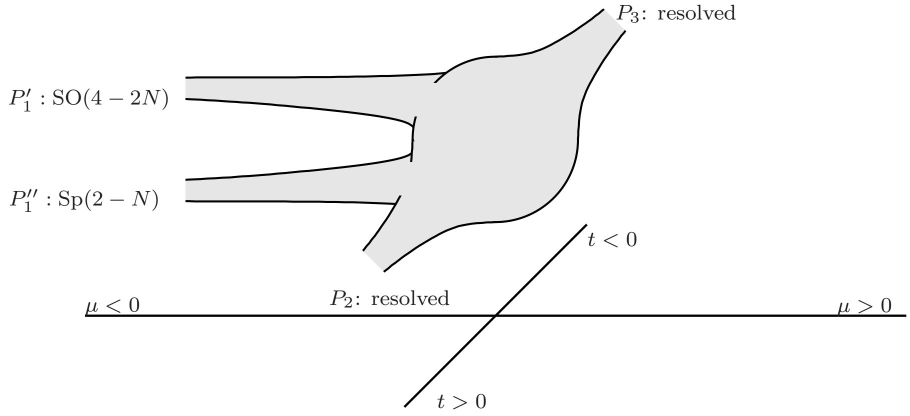

Figure 10: The quantum moduli space for the case $N < 0$.

away (in the covering space) from its anti-D0-brane image, and the ground state of the open string between them is non-tachyonic. (With more than one D0-$\overline{\text{D0}}$ pair in the covering space, a D0 and an anti-D0 from different pairs can approach each other asymptotically, and annihilate.) In a process in which the D0 approaches an O6$^-$-plane, the tachyon will develop and the D0/$\overline{\text{D0}}$ can annihilate. When $N = 0$, this is fine since $\mathbb{Z}_2'$ is broken. When $N > 0$, in which case $\mathbb{Z}_2'$ is globally conserved, this charge is carried away by open strings on the D6-brane describing its separation from the O-plane. (These states are massive after wrapping on S^3, but the D0-brane is much heavier at weak string coupling.)

Let us also note that $\mathbb{Z}_2'$ is preserved in front of an O6$^+$-plane (our case $N = 4$). This is consistent with the fact that the 0–0 tachyon lands in the anti-symmetric representation and is orientifolded out.

5.1.6 $N < 0$

Finally we come back to the regular case. The singularities can be read from the following table:

	P_1'	P_1''	P_2	P_3
η_1	1	-1	0	∞
η_2	$\infty^{2(\tilde N-1)}$	$\infty^{2(\tilde N+1)}$	$0^{4\tilde N}$	1
η_3	$0^{2(\tilde N-1)}$	$0^{2(\tilde N+1)}$	1	$\infty^{4\tilde N}$

where $\tilde N = 2 - N > 2$. And the quantum curve is:

$$\begin{aligned} \eta_2 =& \eta_1^{4\tilde N}(\eta_1-1)^{-2(\tilde N-1)}(\eta_1+1)^{-2(\tilde N+1)}, \\ \eta_3 =& (\eta_1-1)^{2(\tilde N-1)}(\eta_1+1)^{2(\tilde N+1)}. \end{aligned} \tag{5.21}$$

The moduli space consists of a single smooth component as in figure 10.

5.2 Using superpotential

A part of the results of the previous subsection can be obtained also by studying the superpotential proposed by Vafa [15] and the relevant computations in [16].

The superpotential is computed on the branch of the resolved conifold with flux through $\mathbb{RP}^2$. It consists of three parts, coming from the four-form flux, the two-form flux and worldsheet instantons:

$$W = W_{4\,\text{flux}} + W_{2\ \text{flux}} + W_{\text{crosscap}}\,.$$

Let t be the complexified Kähler class of the base $\mathbb{P}^1$ of the resolved conifold. In the present orientifold, the periodicity of the parameter is doubled,

$$t \equiv t + 4\pi i\,. \tag{5.22}$$

The reason is that there exist crosscap diagrams associated with the odd degree maps $S^2 \to \mathbb{P}^1$ which are equivariant with respect to the involution $\Omega : w \to -1/\bar{w}$ on the domain and the anti-holomorphic involution (2.17) on the target. For example, the identity map is such a map and has degree 1. The path-integral weight of such a diagram include odd powers of $e^{-i\text{Im}(t)/2}$. Thus, $z = e^{-t/2}$ is the single valued parameter of the theory. Now, let us describe each term of the superpotential. The contributions from the RR two-form flux through the $\mathbb{RP}^2$ and worldsheet instantons are [15, 16]

$$\begin{aligned} W_{2\ \text{flux}} &= (N-2)\frac{\partial F_0}{\partial t} = -(N-2)Li_2(z^2)\,, \\ W_{\text{crosscap}} &= -4\sum_{m:\text{odd}\geq 1}\frac{e^{-mt/2}}{m^2} = -2(Li_2(z) - Li_2(-z))\,, \end{aligned}$$

where Li_2 is the dilogarithm function. For convenience, some of its properties are collected in appendix B. The contribution from the four-form flux is given by

$$W_{4\,\text{flux}} = -\int F_4 \wedge \omega\,,$$

where ω is the complexified Kähler form of the resolved conifold. According to [15], F_4 has an imaginary part corresponding to the RR four-form and a real NSNS part coming from the failure of the metric to be Calabi-Yau. On a non-compact space it is natural to interpret this formula as follows:

$$W_{4\,\text{flux}} = -\int_M d(\text{Re}(\hat{\Omega}) + iC_3) \wedge \omega = -\int_{\partial M} (\text{Re}(\hat{\Omega}) + iC_3) \wedge \omega\,,$$

where $\hat{\Omega}$ is a suitably normalized form of the holomorphic three-form which is the superpartner of C_3. Evaluating this for the $\mathbb{Z}_2$ quotient of the conifold we obtain

$$W_{4\,\text{flux}} = -\frac{Yt}{2}\,,$$

where Y is the holomorphic volume of the boundary S^3:

$$Y := \int_{S^3_\infty} (\text{Re}(\hat{\Omega}) + iC_3)\,. \tag{5.23}$$

Summing up the three terms, we obtain the total superpotential

$$W = -\frac{Yt}{2} - (N-2)Li_2(z^2) - 2(Li_2(z) - Li_2(-z))\,. \tag{5.24}$$

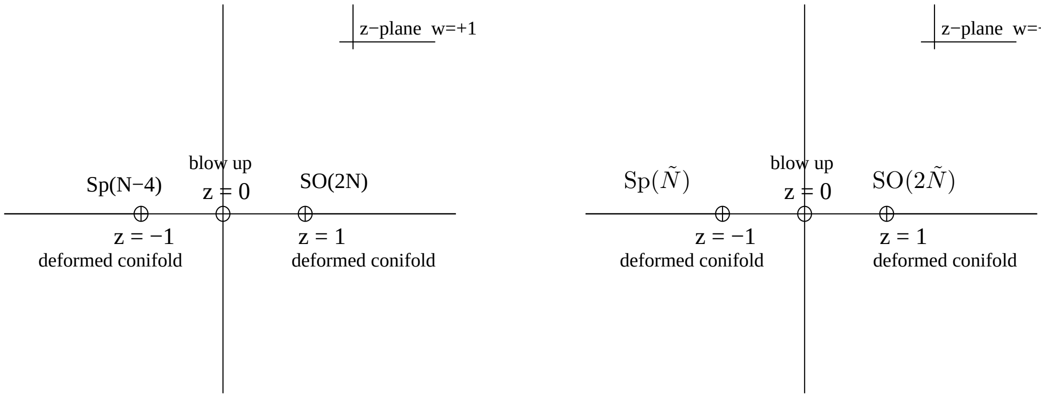

Figure 11: The superpotential for the orientifold system has two branch cuts at $z = \pm 1$ labeling different gauge groups depending on the orientifold action. On the left hand side, the orientifold action with fixed points (orientifold planes). On the right hand side the free orientifold.

Note that the parameters t and Y introduced here are related to the coordinates that we used in the M-theory description of the moduli space as

$$z = \exp\left(-\frac{t}{2}\right) = \eta_1 \,, \qquad \exp(Y) = \eta_3 \,. \tag{5.25}$$

Following [15, 32] we can use this superpotential to find the exact form of the moduli space. In order to have a supersymmetric background we should vary the superpotential with respect to t and find a stationary point. This gives

$$\partial_t W = 0 \Rightarrow Y = \log\Big((z-1)^{-(2N-2)}(z+1)^{-(2N-6)}\Big) \,. \tag{5.26}$$

Using the relation (5.25), we see that this is nothing but the equation describing the component of the moduli space which is smoothly connected to the resolved geometries.

Comparison to 4d gauge theory. The superpotential (5.24) has two branch cuts (see figure 11), starting at $z = \pm 1$. For $z = \pm e^{-\varepsilon}$ with small ε, we have

$$W = -Y\varepsilon - b\varepsilon \log \varepsilon + \cdots \,, \qquad b = \begin{cases} 2N-2 \text{ at } z = 1, \\ 2N-6 \text{ at } z = -1, \end{cases} \tag{5.27}$$

where $+\cdots$ is a power series in ε. When b is positive (*resp.* negative), by the relation (5.26), $\mathrm{Re}(Y)$ diverges to $+\infty$ (*resp.* $-\infty$) as $\varepsilon \to 0$. This is the classical limit where we have a large minimal three sphere $S^3_>$ (*resp.* $S^3_<$) on which a certain number of D6-branes (*resp.* anti-D6-branes) are wrapped. Below we compare this behavior of the superpotential with what we expect from the gauge theory on the sixbranes.

For this purpose, one needs to understand the precise relation of the parameter Y, which can be regarded as the membrane instanton action on a cycle homologous to S^3_∞, and the holomorphic gauge coupling constant $\frac{8\pi^2}{g^2} - i\theta$ of the $4d$ gauge theory on the (anti-)D6-branes wrapped on the minimal three sphere. This was discussed in [14] for the case with O6-plane at S^3. Here we present another argument, using the embedding of

SO($2n$) or Sp(n) into U($2n$) defined by the orientifold projection, which is applicable to the more general systems we are studying. We first note that the instanton number of $4d$ Yang-Mills theory of a simple gauge group G is defined as

$$k = -\frac{1}{8\pi^2}\int_{\mathbb{R}^4} \mathrm{Tr}(F_A \wedge F_A)\,,$$

where Tr is such that the long root has length squared 2, or the trace in the adjoint representation is given by $\mathrm{tr}_{\mathrm{adjoint}}(XY) = 2\check{h}\,\mathrm{Tr}(XY)$, with $\check{h}$ being the dual Coxeter number of G. For the groups SO($2n$) and Sp(n), it is related to the trace of the fundamental representation of U($2n$) by

$$\mathrm{Tr}(F_A \wedge F_A) = \sigma \times \mathrm{tr}_{\mathrm{fund}}(F_{\tilde{A}} \wedge F_{\tilde{A}}), \qquad \sigma = \begin{cases} \frac{1}{2} \text{ for SO}(2n) \\ 1 \text{ for Sp}(n) \end{cases}$$

where $\tilde{A}$ is the U($2n$) gauge field which is obtained from A by the embedding of SO($2n$) or Sp(n) into U($2n$). Thus, one SO($2n$) *instanton corresponds to* two U($2n$) *instantons, while* one Sp(n) *instanton corresponds to* one U($2n$) *instanton.*

First, we consider the case $\mathrm{Re}(Y) \gg 0$ for which the type-IIA geometry is the deformed conifold with an O6-plane at the large minimal three sphere $S^3_>$. Note that this $S^3_>$ is homologous to S^3_∞, and hence Y is the action for one D2-brane wrapped on $S^3_>$. Before orientifold, this D2-brane is wrapped twice on S^3 and corresponds to *two* U($2n$) instantons on $2n$ D6-branes wrapped on S^3. By the remark above, after orientifold, it corresponds to *one* SO($2n$) instanton or *two* Sp(n) instantons. This is indeed the claim in [14]. Thus, the relation of Y and the holomorphic gauge coupling is

$$Y = 2\sigma\left(\frac{8\pi^2}{g^2} - i\theta\right). \tag{5.28}$$

Note that the Chan-Paton factor of the D2-branes on $S^3_>$ is symplectic for O6$^-$ (SO($2n$) on n D6), while it is orthogonal for O6$^+$ (Sp(n) on n D6). Thus, the number of D2-branes for O6$^-$ must be even in the double cover. For O6$^+$, on the other hand, a "half" D2-brane (one in the cover) is allowed and corresponds to *one* Sp(n) Yang-Mills instanton with instanton factor $e^{-Y/2}$. In terms of $S = 2\sigma\varepsilon$, the superpotential (5.27) can be written as

$$W = -\left(\frac{8\pi^2}{g^2} - i\theta\right)S - \frac{b}{2\sigma}S\log S + \text{power series in } S\,. \tag{5.29}$$

If we identify $z = 1$ as the large $S^3_>$ limit with O6$^-$-plane and $z = -1$ as the large $S^3_>$ limit with O6$^+$-plane, the coefficient of the $-S\log S$ term is

$$\frac{b}{2\sigma} = \begin{cases} 2N - 2 & \text{at } z = 1 \\ N - 3 & \text{at } z = -1\,, \end{cases}$$

which are the dual Coxeter number of the groups SO($2N$) and Sp($N-4$) respectively. Then, (5.29) is exactly the Veneziano-Yankielowicz superpotential, up to a power series in S, for the gauge group SO($2N$) for $z = 1$ and Sp($N-4$) for $z = -1$.

Let us next consider the case $\mathrm{Re}(Y) \ll 0$ which corresponds to the free orientifold of the deformed conifold with large minimal three sphere $S^3_<$. The cycle $S^3_<$ is also homologous to S^3_∞, and Y corresponds to the action for D2-brane wrapped twice on $\mathbb{RP}^3 = S^3_</\mathbb{Z}_2$. This again corresponds to *two* $\mathrm{U}(2n)$ instantons on $2n$ D6-branes wrapped on $S^3_<$ before orientifold, and thus to *one* $\mathrm{SO}(2n)$ or *two* $\mathrm{Sp}(n)$ instanton after orientifold. On anti-D6-branes, instantons with positive instanton numbers correspond to *anti*-D2-branes. Thus the relation of Y and the holomorphic gauge coupling on the anti-D6-branes is

$$Y = -2\sigma\left(\frac{8\pi^2}{g^2} - i\theta\right). \tag{5.30}$$

As in the $\mathrm{Re}(Y) \gg 0$ case, the minimal instanton factor is e^Y for $\mathrm{SO}(2n)$ and $e^{Y/2}$ for $\mathrm{Sp}(n)$. In terms of $S = -2\sigma\varepsilon$, the superpotential is

$$W = -\left(\frac{8\pi^2}{g^2} - i\theta\right)S + \frac{b}{2\sigma}S\log S + \text{power series in } S\,.$$

This is exactly the expected Veneziano-Yankielowicz superpotential if we identify $z = 1$ (*resp.* $z = -1$) as the large $S^3_<$ limit with SO (*resp.* Sp) gauge group. Indeed, under this identification, the coefficient of the $-S\log S$ term is

$$-\frac{b}{2\sigma} = \begin{cases} -2N + 2 = 2\tilde{N} - 2 & \text{at } z = 1 \\ -N + 3 = \tilde{N} + 1 & \text{at } z = -1\,, \end{cases}$$

which are the dual Coxeter number of the groups $\mathrm{SO}(2\tilde{N})$ and $\mathrm{Sp}(\tilde{N})$ respectively, where $\tilde{N} = 2 - N$ is half the number of anti-D6-branes.

Summary on the component including the resolved conifolds. By the above comparison with $4d$ gauge theory along with the information about the expected classical gauge groups in the various limits, we arrive at the following consistent set of rules for reading the classical configurations from the superpotential analysis depending on the RR charge and the particular point in the moduli space:

A) at the point $z = -1$ there is an Sp group, the number of vacua (or better said the instanton counting) is $2N - 6$. The gauge group will depend on the number of vacua:

- A.1) if $2N - 6 > 0$ the gauge group is $\mathrm{Sp}(N - 4)$. The classical description is an $\mathrm{O6}^+$ with $N - 4$ D6-branes.
- A.2) if $2N - 6 = 0$ there is no classical limit at $z = -1$.
- A.3) if $2N - 6 < 0$ the gauge group is $\mathrm{Sp}(2 - N)$. The classical description is a free orientifold with $4 - 2N$ anti-D6-branes.

B) At the point $z = 1$ the structure is similar:

- B.1) if $2N - 2 > 0$ the gauge group is $\mathrm{SO}(2N)$. The classical description is an $\mathrm{O6}^-$ with N D6-branes.

B.2) if $2N - 2 = 0$ there is no classical limit at $z = 1$.

B.3) if $2N - 2 < 0$ the gauge group is $\mathrm{SO}(4 - 2N)$. The classical description is the free orientifold with $4 - 2N$ anti-D6-branes.

These rules allow us to classify the different possibilities depending on the RR charge:

(i) For $N > 3$ there are four classical points:

$\mathrm{O6}^-$ and N D6, gauge group $\mathrm{SO}(2N)$
$\mathrm{O6}^+$ and $N - 4$ D6, gauge group $\mathrm{Sp}(N - 4)$
two free orientifolds of the resolved conifold

(ii) For $N = 3$ there are three classical points (the curve is the same as the SU(4) curve):

$\mathrm{O6}^-$ and N D6, gauge group $\mathrm{SO}(6) \sim \mathrm{SU}(4)$
two free orientifolds of the resolved conifold

(iii) For $N = 2$ there are four classical points (and others sitting on the other branch):

$\mathrm{O6}^-$ and N D6, gauge group SO(4)
free orientifold of deformed conifold, gauge group Sp(0) (= nothing)
two free orientifolds of the resolved conifold

(iv) For $N = 1$ there are 3 classical points (two others sitting on the branch with an infrared U(1)):

free orientifold and 2 anti-D6 on $\mathbb{RP}^3$, gauge group $\mathrm{Sp}(1) = \mathrm{SU}(2)$
two free orientifolds of the resolved conifold

(v) For $N = 0$ there are 4 classical points (and a few others on the other branch):

free orientifold and 4 anti-D6 on $\mathbb{RP}^3$, gauge group SO(4)
free orientifold and 4 anti-D6 on $\mathbb{RP}^3$, gauge group Sp(2)
two free orientifolds of the resolved conifold

(vi) For $N < 0$ there are four classical points:

free orientifold and $4 - 2N$ anti-D6 on $\mathbb{RP}^3$, gauge group $\mathrm{SO}(2(2 - N))$
free orientifold and $4 - 2N$ anti-D6 on $\mathbb{RP}^3$, gauge group $\mathrm{Sp}(2 - N)$
two free orientifolds of the resolved conifold

6. Exact branch structure for $N = 1$

In this section we wish to present a detailed analysis of the branch structure for the $N = 1$ case of the previous section from various dual pictures. As advertised, an exact description of the quantum moduli space is obtained by moving to a mirror type-IIB theory.

We consider the strong coupling limit of the original type IIA, for which the relevant M-theory geometry is asymptotically a cone over $(S^3 \times S^3)/D_3'$. Here the group $D_3' \simeq \mathbb{Z}_4$ is introduced in section 3.3 and acts on the triplet of SU(2) matrices as

$$(g_1, g_2, g_3) \to (i\tau_2 g_1, -g_2, g_3)\,.$$

Dimensional reduction along the orbit of the U(1) action $(g_1, g_2, g_3) \to (e^{i\alpha\tau_3} g_1, g_2, g_3)$ brings the system back to the original type IIA. If we reduce instead along a diagonal S^1,

$$(g_1, g_2, g_3) \to (e^{i\alpha\tau_2} g_1, e^{i\alpha\tau_2} g_2, e^{i\alpha\tau_3} g_2)\,,$$

the resulting type-IIA configuration is a partially blown-up orbifold $\mathbb{C}^3/D_3'$ with a D6-brane of topology $\mathbb{R}^2 \times S^1$. Here the generator of D_3' acts on the coordinates of $\mathbb{C}^3$ as

$$(z_1, z_2, z_3) \to (-z_1, iz_2, iz_3)\,. \tag{6.1}$$

The whole system admits a GLSM description.

The main advantage of this framework is that one can read off the quantum moduli space rather directly by moving to the mirror IIB description. This chain of dualities was used in [32] to study the geometric transition of D6-branes wrapped on the S^3 of deformed conifold, and also applied in [24] to study its $\mathbb{Z}_2$ orbifold.

GLSM description. Consider the edge vectors for the toric fan of $\mathbb{C}^3$,

$$v_1 = (1,0,0)\,, \qquad v_2 = (0,1,0)\,, \qquad v_3 = (0,0,1)\,,$$

generating the lattice $\mathbb{Z}^3$ of torus actions. The orbifolding makes the lattice finer by the inclusion of an extra generator $\rho = (-\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$. The fan for Calabi-Yau resolution of orbifold singularity is given by including the lattice points $v_4 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ and $v_5 = (0, \frac{1}{2}, \frac{1}{2})$ as new edge vectors and subdividing the fan. See figure 12.

The GLSM consists of five chiral fields $z_{1,\cdots,5}$ and $\mathrm{U}(1)^2$ gauge symmetry. The U(1) charges which span the Mori cone are given by

$$Q^1 = (1,0,0,-2,1)\,, \qquad Q^2 = (0,1,1,0,-2)\,, \tag{6.2}$$

so that the fully blown-up phase is given by the D-term equations

$$|z_1|^2 - 2|z_4|^2 + |z_5|^2 = r^1 > 0\,, \qquad |z_2|^2 + |z_3|^2 - 2|z_5|^2 = r^2 > 0\,. \tag{6.3}$$

In the orbifold phase with negative FI parameters the fields z_4, z_5 acquire vev and break the gauge group down to $\mathbb{Z}_4$, which acts (z_1, z_2, z_3) to $(-z_1, iz_2, iz_3)$. It is useful to draw the toric skeleton diagram describing the base polytope of T^3 fibration.

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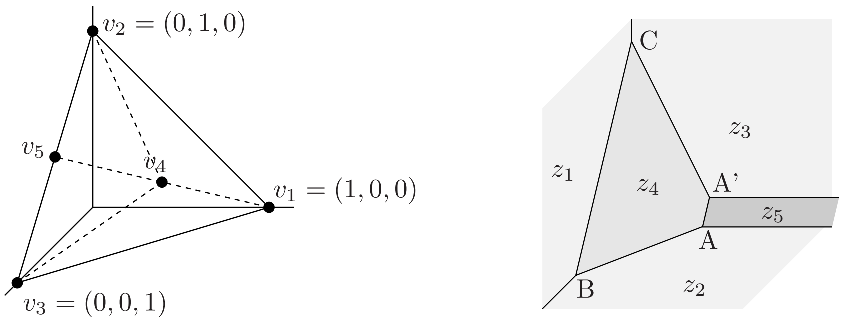

Figure 12: (Left) toric fan for the orbifold $\mathbb{C}^3/\mathbb{Z}_4$, (Right) skeleton for the fully blown-up phase.

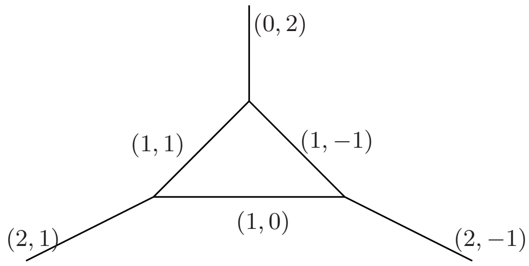

Figure 13: The fivebrane web which is dual to a partial resolution of the orbifold $\mathbb{C}^3/\mathbb{Z}_4$.

In addition to the Kähler parameters, one has to specify the location of D6-brane which projects to a half-line ending on a one-dimensional face of the toric polytope. As was discussed in [24] for related models, one cannot choose the Kähler parameters and the location of the D6-brane independently, because of a superpotential generated by the D6-brane. Also, by looking into the asymptotics one finds that the resolution mode corresponding to z_5 should not be turned on. In other words, in the skeleton diagram the points A, A' should coincide.

Three-dimensional five-brane web. One can understand the effect of superpotential semiclassically by relating our GLSM picture to a type-IIB five-brane web. Under a suitable choice of basis for the charge, the partial blowup of our orbifold $\mathbb{C}^3/\mathbb{Z}_4$ is mapped to the two-dimensional web of figure 13. The D6-brane in GLSM picture turns into another five-brane leg carrying a new kind of charge ending on a leg of the web. Addition of such a five-brane makes the web three-dimensional. The supersymmetry condition for the resulting web constrains the allowed locations of the D6-brane endpoint for each choice of Kähler parameter.

The charges of each leg of the three-dimensional web can be obtained by regarding the G_2 holonomy manifolds as T^3 fibrations, and analyzing the locus of degenerate fiber in the base. Since these charges are relevant in calculating the superpotential, let us go back to M-theory and calculate them explicitly. As the T^3 we take the orbit of $\mathrm{U}(1)^3$ action

$$[\alpha_1, \alpha_2, \alpha_3]: \quad (g_1, g_2, g_3) \;\mapsto\; (e^{2\pi i\alpha_1\tau_2} g_1, e^{2\pi i\alpha_2\tau_2} g_2, e^{2\pi i\alpha_3\tau_2} g_3)\,, \tag{6.4}$$

modulo identification

$$\begin{aligned}
[\alpha_1,\alpha_2,\alpha_3] &\sim [\alpha_1,\alpha_2,\alpha_3] + \left[-\frac{1}{4},-\frac{1}{2},0\right] \\
&\sim [\alpha_1,\alpha_2,\alpha_3] + \left[\frac{1}{2},0,0\right] \\
&\sim [\alpha_1,\alpha_2,\alpha_3] + \left[\frac{1}{2},\frac{1}{2},\frac{1}{2}\right] .
\end{aligned} \tag{6.5}$$

In the following we will also use the coordinate $(X,\tilde{X}) \equiv (g_2 g_1^{-1}, g_3 g_1^{-1})$ of $S^3 \times S^3$.

The classical moduli space consists of the following branches. A free orientifold of deformed conifold with two D6-branes corresponds to g_1 filled in, and the orientifold of resolved conifold with flux correspond to either g_2 or g_3 filled in. The $\mathrm{O6}^-$+D6 configuration corresponds to the geometry with the following S^1 shrinking at $r=0$,

$$(X,\tilde{X}) \mapsto (Xe^{2\pi i\alpha\tau_2}, \tilde{X}e^{-2\pi i\alpha\tau_2}) . \tag{6.6}$$

The cross-section of the seven-manifold at any finite r is $S^3 \times S^3$. It can be viewed as a T^3 fibration over S^3, and a section is given by

$$(X,\tilde{X}) = \left(e^{i\theta\tau_1/2}e^{i\varphi\tau_2}, e^{i\tilde{\theta}\tau_1/2}\right), \qquad (0\le\theta\le\pi\,,\ 0\le\tilde{\theta}\le\pi\,,\ 0\le\varphi\le 2\pi) .$$

At four special points on the base, the fiber T^3 has a vanishing one-cycle labeled by the ratio of $[\alpha_1,\alpha_2,\alpha_3]$,

$$\begin{aligned}
(\theta,\tilde{\theta}) = (0,0) &\ \cdots\ [\alpha_1,\alpha_2,\alpha_3] = \left[+\frac{1}{2},+\frac{1}{2},+\frac{1}{2}\right], \\
(\theta,\tilde{\theta}) = (0,\pi) &\ \cdots\ [\alpha_1,\alpha_2,\alpha_3] = \left[-\frac{1}{2},-\frac{1}{2},+\frac{1}{2}\right], \\
(\theta,\tilde{\theta}) = (\pi,0) &\ \cdots\ [\alpha_1,\alpha_2,\alpha_3] = \left[-\frac{1}{2},+\frac{1}{2},-\frac{1}{2}\right], \\
(\theta,\tilde{\theta}) = (\pi,\pi) &\ \cdots\ [\alpha_1,\alpha_2,\alpha_3] = \left[+\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right].
\end{aligned} \tag{6.7}$$

Thus we find four half-lines of degenerate fiber. There are additional loci of degenerate fiber at $r=0$. For the classical branch with g_1 filled in one finds a line segment of degenerate $[1,0,0]$ one-cycle at $r=0$, and similarly for the other two branches where g_2 or g_3 are filled in. For the last branch, one finds that the vanishing S^1 given in (6.6) lies along the T^3 fiber when

$$\begin{aligned}
\theta = 0 &\ \cdots\ [\alpha_1,\alpha_2,\alpha_3] = \left[+\frac{1}{4},+\frac{1}{2},0\right], \\
\theta = \pi &\ \cdots\ [\alpha_1,\alpha_2,\alpha_3] = \left[+\frac{1}{4},-\frac{1}{2},0\right], \\
\tilde{\theta} = 0 &\ \cdots\ [\alpha_1,\alpha_2,\alpha_3] = \left[-\frac{1}{4},0,-\frac{1}{2}\right],
\end{aligned}$$

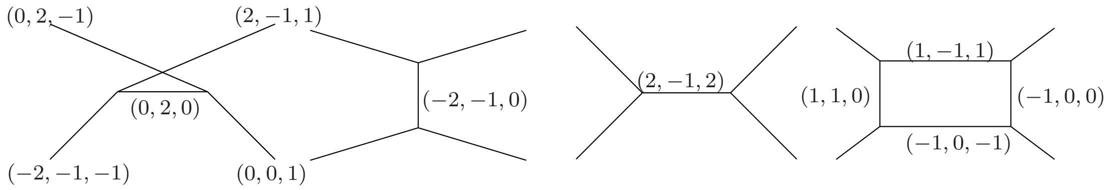

Figure 14: The fivebrane web corresponding to the four families of G_2 holonomy spaces.

$$\tilde{\theta} = \pi \cdots [\alpha_1, \alpha_2, \alpha_3] = \left[-\frac{1}{4}, 0, +\frac{1}{2}\right]. \tag{6.8}$$

By re-labeling the shrinking one-cycle in terms of the fundamental periods given in (6.5), one finds that the four half-lines of (6.7) are labeled by

$$(0,0,1)\,, \qquad (2,-1,1)\,, \qquad (-2,-1,-1)\,, \qquad (0,2,-1)\,. \tag{6.9}$$

We can identify them with the charges of semi-infinite five-brane legs forming three-dimensional webs. Indeed, the first leg corresponds to the D6-brane while the other three project to the legs of two-dimensional web of figure 13. The four families of G_2 holonomy spaces thus correspond to the three-dimensional webs summarized in figure 14. All these four webs are rigid apart from overall rescaling.

IIB mirror. The mirror of this GLSM is given by a LG model with superpotential

$$F(x,y) = y^2 + e^{\frac{t_1}{2}+\frac{t_2}{4}} xy + x^3 + e^{\frac{t_2}{2}} x^2 + x\,. \tag{6.10}$$

Here (x, y) are $\mathbb{C}^\times$-valued chiral superfields, and t_1, t_2 are the Kähler parameters in the GLSM. The spacetime theory is a type-IIB superstring on a local Calabi-Yau manifold

$$\xi\eta = F(x,y)\,, \tag{6.11}$$

with a D5-brane wrapping a holomorphic curve $\eta = F(x, y) = 0$. We denote by $(x_{\mathbf{D}}, y_{\mathbf{D}})$ its position on the curve $\Sigma : F(x, y) = 0$.

The parameter t_2 is fixed by requiring the curve Σ to have only three punctures. There seems to be two choices $e^{\frac{t_2}{2}} = \pm 2$, but they should lead to the same structure for the moduli space so we choose $e^{\frac{t_2}{2}} = -2$. The remaining normalizable parameter $s \equiv -ie^{\frac{t_1}{2}}/\sqrt{2}$ in the curve,

$$\Sigma: \quad y^2 - 2sxy + x^3 - 2x^2 + x = 0\,, \tag{6.12}$$

is stabilized by the superpotential generated by the D5-brane.

$$W(s) = \int_{\mathbf{A}}^{\mathbf{D}} + \int_{\mathbf{C}}^{\mathbf{B}} \log y \cdot d\log x\,. \qquad (\mathbf{A}(1,0),\ \mathbf{B}(0,0),\ \mathbf{C}(\infty,\infty))\,. \tag{6.13}$$

This integration contour is chosen from the observation that the four semi-infinite five-brane legs all have non-zero "third" charge, which we interpret as the presence of D5 or $\overline{\text{D5}}$-branes at three punctures **A, B, C** of the curve.

Note that the moduli space of curves Σ is the complex s-plane modulo identification $s \sim -s$, since $s \sim e^{\frac{t_1}{2}}$. Indeed, the sign flip of s can be absorbed by the sign flip of y. The good modulus of the curve is therefore s^2.

For generic s the curve Σ is of genus one. The degeneration to genus zero occurs at $s = 0$ or $s = \pm 2i$. For genus one curve, the modulus s is related to the position of $\mathbf{D}$ because of the superpotential. Using $\hat{y} = y - sx$ we rewrite the equation for the curve as

$$\hat{y}^2 + x^3 - (2 + s^2)x^2 + x = 0\,. \tag{6.14}$$

The F-term condition reads

$$\int_{\overline{\mathrm{D5}}=\mathbf{A},\mathbf{C}}^{\mathrm{D5}=\mathbf{B},\mathbf{D}} \frac{dx}{\hat{y}} = 0\,. \tag{6.15}$$

One can solve this equation by fixing s in such a way that there is a meromorphic function on Σ with simple zeroes at $\mathbf{B},\ \mathbf{D}$ and simple poles at $\mathbf{A}, \mathbf{C}$.

$$\begin{vmatrix} 1 & x_{\mathbf{A}} & -\hat{y}_{\mathbf{A}} \\ 1 & x_{\mathbf{B}} & \hat{y}_{\mathbf{B}} \\ 1 & x_{\mathbf{D}} & \hat{y}_{\mathbf{D}} \end{vmatrix} = \begin{vmatrix} 1 & 1 & s \\ 1 & 0 & 0 \\ 1 & x_{\mathbf{D}} & \hat{y}_{\mathbf{D}} \end{vmatrix} = s x_{\mathbf{D}} - \hat{y}_{\mathbf{D}} = 0 \quad \Rightarrow \quad (x_{\mathbf{D}}, y_{\mathbf{D}}) = (1, 2s)\,. \tag{6.16}$$

The moduli space for the case $N = 1$ is made of $g = 0$ and $g = 1$ branches. The $g = 1$ branch is the moduli space of genus one curves with three punctures, and is the complex s^2-plane as explained above. It has special points $s^2 = 0, -4$ and infinity. At $s = 2i$ the curve degenerates to genus zero,

$$\Sigma: \quad y^2 - 4ixy + x^3 - 2x^2 + x = 0\,. \tag{6.17}$$

This curve itself is regarded as the $g = 0$ branch. Various semi-classical points on moduli space are identified with those in the conifold picture as follows:

branch	point	gauge group	description
$g = 0$	$\mathbf{A}$	Sp(1)	deformed$_{(\mu<0)}$, with 2D6 on $\mathbb{RP}^3$
$g = 0$	$\mathbf{B,C}$	none	resolved, with (-1) flux
$g = 1$	$s^2 = 0$	O(2)	deformed$_{(\mu<0)}$, with 2D6 on $\mathbb{RP}^3$
$g = 1$	$s^2 = \infty$	O(2)	deformed$_{(\mu>0)}$, with O6$^-$+D6 on S^3

By a suitable identification of the variables η_i with (x, y) one can identify the z-plane of section 5.2 with the $g = 0$ branch here, and also see the expected behavior of η_i's on the other branch which can be identified with the s^2-plane. The key fact is that the membrane instantons wrapping on three-cycles $\{2E_1', E_2', E_3'\}$ of section 5.1 turn into disc instantons bounded by the D6-brane. Matching of their volumes gives

$$\eta_1^2 \sim e^{(|z_2|^2 - |z_3|^2)/2} \sim x\,, \qquad \eta_2 \sim e^{|z_1|^2 - |z_3|^2} \sim x^3 y^{-2}\,, \qquad \eta_3 \sim e^{|z_2|^2 - |z_1|^2} \sim x^{-1} y^2, \tag{6.18}$$

where we used the standard mirror identification

$$e^{-|z_1|^2} : e^{-|z_2|^2} : e^{-|z_3|^2} : e^{-|z_4|^2} : e^{-|z_5|^2} \sim y^2 : x^3 : x : xy : x^2\,, \tag{6.19}$$

and "$\sim$" expresses the identification up to phase. Under this identification, the $g = 0$ curve (6.17) precisely agrees with the relation (5.18) among η_i on the branch of mass-gapped vacua obtained in the previous section. Note that the η_1 defined as

$$\eta_1 = i\left(\frac{y - 2ix}{1 + x}\right) \tag{6.20}$$

is single valued on Σ and it indeed squares to x on Σ. Note also that, although the $g = 0$ curve (6.17) has a double point, the singularity is blown up on generic points of $g = 0$ branch and the η_i indeed take two different values there. The functional form of η_i's on the $g = 1$ branch is obtained simply by substituting $(x, y) = (1, 2s)$ into (6.18):

$$\eta_1 = 1\,, \qquad \eta_2 = \frac{1}{4s^2}\,, \qquad \eta_3 = 4s^2\,. \tag{6.21}$$

This agrees with the expectation in section 5.1.4 that the branch of vacua with infrared U(1) is a cylinder $\eta_2\eta_3 = 1$.

7. Other cases

We wish to briefly comment on the possibility of μ-transitions at the conifold when the orientifold action is in one of the other classes discussed in section 2. It is clear that it will be much harder to find the associated G_2 holonomy metrics. Nevertheless, qualitative considerations similar to the ones we sometimes used above give good indications when we should expect a μ-transition.

7.1 Case (1)$\leftrightarrow$(3)

The first thing to notice in cases (1) and (3) is that the O-plane intersects the compact three-cycle of the deformed conifold. This creates flux that cannot escape to infinity, and we should cancel the flux by wrapping a fixed number of D6-branes. A simple class of cycles to wrap branes around is the fixed point set of some anti-holomorphic involution which can be different from the one used to define the orientifold, but must be in the same class to preserve supersymmetry.

In case (1), the O6-plane is the fixed point locus of the involution $z \mapsto M_O\bar{z}$, where M_O is the orthogonal matrix diag$(-1, 1, 1, 1)$. The O6-plane is topologically $S^2 \times \mathbb{R}$ and the intersection number with the $S^3_>$ is two. If this is a standard O6$^-$-plane with negative twice the charge of a D6-brane (as measured in the quotient space), this means that we need to wrap a D6-brane configuration that intersects the compact cycle exactly four times (namely, an invariant configuration in the covering space intersecting $S^3_>$ in eight points). We know of supersymmetric cycles intersecting the $S^3_>$ twice: The fixed point locus of $z \mapsto M\bar{z}$, where M is another orthogonal matrix with eigenvalues $(-1, 1, 1, 1)$ (generally distinct from M_O). We can write $M = UM_OU^T$, where U is an element of SO(4), and two U's give the same M if they differ by an element of SO(3). In the covering space, the possible M's live in SO(4)/ SO(3) $\simeq S^3$, and we need two (pairs of) such cycles to cancel the charge. Orientifolding maps the brane associated with U to the brane associated with $M_O U M_O$, which corresponds to acting on S^3 as the element $\mathrm{diag}(1,-1,-1,-1)$ of SO(4). In the orientifold, the space of possible brane wrappings is therefore the symmetric product $\mathcal{M}_-^{(1)} \simeq \mathcal{S}^2(\mathcal{L})$, where $\mathcal{L} = S^3/\mathbb{Z}_2$ with the given action of $\mathbb{Z}_2$.

Note that if the O-plane is an $\mathrm{O6}^-$, we cannot wrap branes on fixed point loci of involutions with three negative eigenvalues, since those would preserve the opposite supersymmetry, and we cannot wrap anti-branes if we are to cancel the charge.

On the other hand, if the orientifold plane is an $\mathrm{O6}^+$ with positive twice the charge of a D6-brane, we need anti-branes to cancel the charge, and those are most conveniently wrapped on fixed point loci of $z \mapsto M\bar{z}$, where M is an orthogonal matrix with eigenvalues $(-1,-1,-1,1)$. As we recall, this gives a cycle with two disconnected components, each of which is a copy of $\mathbb{R}^3$ and intersects the $S^3_>$ once. In that case, we have the choice of four such cycles, each of which is again parameterized by the choice of $\mathcal{L} \simeq S^3/\mathbb{Z}_2 \simeq \big(\mathrm{SO}(4)/\mathrm{SO}(3)\big)/\mathbb{Z}_2$. Thus, $\mathcal{M}_+^{(1)} \simeq \mathcal{S}^4(\mathcal{L})$.

In case (3), the situation is reversed: If the O-plane on $\mathbb{R}^3 \cup \mathbb{R}^3$ is an $\mathrm{O6}^-$, we can wrap four D6-branes on four different copies of $\mathbb{R}^3$, while with an $\mathrm{O6}^+$, we can wrap two anti-D6-branes on $S^2 \times \mathbb{R}$. We have $\mathcal{M}_-^{(3)} \simeq \mathcal{S}^4(\mathcal{L})$, and $\mathcal{M}_+^{(3)} \simeq \mathcal{S}^2(\mathcal{L})$, where, importantly, $\mathcal{L} \simeq S^3/\mathbb{Z}_2$ is the same quotient as before.

It is worthwhile to point out that the parameters associated with the brane wrappings are not on the same footing as the parameter associated with the size of the S^3. The latter parameter, although not normalizable in the non-compact geometry, is still localized and will survive embedding in a compact model. The data parameterizing the positions of the branes, on the other hand, is completely fixed at infinity. Not even their complexification can be determined before embedding in a compact model.

As a consequence, in addition to fixing the sign of the O-plane, we should fix a point $(M_1, M_2) \in \mathcal{M}_-^{(1)}$ when we attempt to take case (1) through a μ transition to case (3). In so doing, we can see from the matching of the D-brane configuration at infinity that we end up with a configuration in $\mathcal{M}_-^{(3)}$ of the special form (M_1, M_1, M_2, M_2). Conversely, starting from $(M_1, M_2, M_3, M_4) \in \mathcal{M}_+^{(1)}$, we can match with a point in $\mathcal{M}_-^{(3)}$ only for $M_1 = M_3$, $M_2 = M_4$.

These constraints allow us to predict a smooth μ-transition only for this subset of D-brane configurations. We are not able to determine the fate of the other configurations as μ goes to zero.

Another point that remains unclear is whether there will be a point of enhanced gauge symmetry in the moduli space. As we have explained before, the orientifolds in the classes (1) and (3) do not admit the resolved conifold. Naively, we can say that the orientifold is projecting out the scalars of the $\mathcal{N} = 2$ vectormultiplet that was associated with the $\mathbb{P}^1$ of the resolved conifold (whereas in cases (0), (2) and (4) we are projecting out the $\mathcal{N} = 1$ vector). One possible conclusion is that the $\mathcal{N} = 1$ vector half of the $\mathcal{N} = 2$ vectormultiplet, which is broken by the hypermultiplet vev at a generic point, reappears at the singular conifold. However, it is not clear that such a naive argument will survive a more careful treatment.

7.2 Case (2)

Now the O-plane locus is $S^1 \times \mathbb{R}^2$ and has zero intersection with the compact three-cycle. We can then wrap supersymmetrically D6-branes on the compact three-cycle and get a dynamical gauge theory in four dimensions. This is very similar to the case (0)/(4) that we have focused on before. We can not find an M-theory lift, but since the small resolution of the conifold is allowed, we can apply the Vafa superpotential method to predict the structure of a part of the quantum parameter space.

Assume that we want to wrap $2N$ D6-branes on $S^3_>$. By the computation of the period integrals in section 2.2, this will preserve the same supersymmetry as an anti-O6-plane wrapped on the noncompact cycle $S^1 \times \mathbb{R}^2$. If this O-plane is an anti-O6$^-$ (with positive charge), the gauge group is $\mathrm{Sp}(N)$. This gauge group type follows because the relative codimension between O-plane and D-brane is 4.

A configuration with the same supersymmetry and charge at infinity is obtained by wrapping $2N+4$ branes, or rather $2\tilde{N} = -2N-4$ anti-D6-branes, on $S^3_<$. Here, the 4 comes again from the jump (2.14) in the class of the fixed point locus, multiplied by the charge of the anti-O6$^-$-plane. Note again that this is supersymmetric and gives the gauge group $\mathrm{Sp}(\tilde{N})$ in front of the anti-O6$^-$-plane because of the lower-dimensional intersection.

We see that N and $\tilde{N}$ are never both non-negative at the same time, so that we do not expect a μ-transition to be possible for any N. In fact, when $N=-1$, also $\tilde{N}=-1$, so this value of the flux does not admit a deformed conifold for the fixed supersymmetry.

Meanwhile, if the O-plane is an anti-O6$^+$, and we wrap $2N$ D6-branes on $S^3_>$, the gauge group will be $\mathrm{SO}(2N)$. Going through the μ-transition, we could also wrap $2N-4$ D6-branes, or rather $2\tilde{N} = -2N+4$ anti-D6-branes on $S^3_<$, with gauge group $\mathrm{SO}(2\tilde{N})$. The jump by -4 is the value familiar from the case (0)/(4), and as in that case, we expect a μ-transition to be possible for $N=0,1,2$, but no other values of N.

Let us now figure out the moduli space using the superpotential.

The moduli space. To start with, we discuss the right parameter of the moduli space near the resolved conifold points. Let t be as before the complexified Kähler class of the $\mathbb{P}^1$ of the resolved conifold. Unlike in the cases (0) and (4) considered in section 5.2 (see (5.22)), the periodicity in the present case is the same as the one before the orientifold:

$$t \equiv t + 2\pi i\,, \tag{7.1}$$

so that the single valued coordinate of the parameter space is e^{-t}. This is because the worldsheet diagram must always have even powers of $\exp(i\mathrm{Im}(t)/2)$. Namely, there is no odd degree smooth map of the worldsheet S^2 to the target $\mathbb{P}^1$ compatible with the involution $\Omega : w \to -1/\bar{w}$ on the domain and the involution τ given by (2.18) on the target. This can be shown as follows. Let $X : S^2 \to \mathbb{P}^1$ be such a map, where the compatibility means $\tau \circ X \circ \Omega = X$. Choosing a Kähler form ω of $\mathbb{P}^1$ of volume 1, the degree is defined as $d = \int_{S^2} X^*\omega$. Let us decompose S^2 as a union of the upper and lower hemi-spheres $S^2 = H_+ \cup H_-$ which are oriented such that $[S^2] = H_+ + H_-$ and $\Omega(H_+) = -H_-$. Using

$\tau^*\omega = -\omega$, one can express the degree as

$$
\begin{aligned}
d &= \int_{H_+} X^*\omega + \int_{H_-} X^*\omega = \int_{H_+} X^*\omega - \int_{H_+} \Omega^* X^*\omega \\
&= \int_{H_+} (X^*\omega + \Omega^* X^* \tau^* \omega) = 2\int_{H_+} X^*\omega \,.
\end{aligned}
$$

The idea is to show that $\int_{H_+} X^*\omega$ is an integer. Note that d is an integer and thus $\int_{H_+} X^*\omega$ is deformation invariant, as long as $X : H_+ \to \mathbb{P}^1$ extends to an equivariant map of S^2 to $\mathbb{P}^1$. Extension to S^2 is possible if and only if the restriction to the boundary $X : \partial H_+ \to \mathbb{P}^1$ is equivariant. It is always possible to shrink this loop to a constant map to a τ-fixed point, keeping the equivariance all the way. Once this is done, we obtain a map $X : H_+/\partial H_+ \to \mathbb{P}^1$, which is a map between two spheres. $\int_{H_+} X^*\omega$ is its degree and thus is an integer.

We first consider the case of an anti-O6$^-$ and $2N \geq 0$ D6-branes on $S^3_>$ or $2\tilde{N} = -2N - 4 \geq 0$ anti-D6-branes on $S^3_<$. The computation is done in the orientifold of the resolved conifold with certain RR two-form flux through $\mathbb{RP}^2$. The flux is N since the O-plane does not contribute to the two-form flux through $\mathbb{RP}^2$ (the O-plane is still there on the resolved side). The superpotential again has three terms corresponding to the three origins: four-form flux, two-form flux and worldsheet instantons. The result is

$$
\begin{aligned}
W &= -\int F_4 \wedge \omega + N\frac{\partial F_0}{\partial t} - 4 \sum_{m:\,\text{even}>0} \frac{e^{-mt/2}}{m^2} \\
&= -\frac{Yt}{2} - N Li_2(e^{-t}) - 2(Li_2(e^{-t/2}) + Li_2(-e^{-t/2})) \\
&= -\frac{Yt}{2} - (N+1) Li_2(e^{-t})\,.
\end{aligned}
\tag{7.2}
$$

The parameter Y is again defined by

$$
Y := \int_{S^3_\infty} (\mathrm{Re}\,\hat{\Omega} + iC_3)\,.
$$

The sum over m in the crosscap part is over even integers because only even degree maps are compatible with the present orientifold, as remarked above. Solving $\partial_t W = 0$, we find the equation determining the moduli space

$$
e^{-Y} = (1 - e^{-t})^{2N+2}\,. \tag{7.3}
$$

It is a copy of the Riemann sphere with three marked points — $(e^{-Y}, e^{-t}) = (1,0)$, (∞,∞), $(0,1)$ for $N \geq 0$, while $(e^{-Y}, e^{-t}) = (1,0)$, $(0,\infty)$, $(\infty,1)$ for $N \leq -2$. In either case, the superpotential has a branch cut at the last marked point, $e^{-t} = 1$, at which it behaves as follows:

$$
W = -\frac{Yt}{2} - (N+1)t\log t + \cdots\,. \tag{7.4}
$$

If $N \geq 0$, the point $(e^{-Y}, e^{-t}) = (0,1)$ corresponds to a large minimal three sphere $S^3_>$ with $2N$ D6-branes supporting an $\mathrm{Sp}(N)$ gauge field. The relation of the parameter Y and the holomorphic gauge coupling $(\frac{8\pi^2}{g^2} - i\theta)$ on the D6-brane can be determined following the argument given in section 5.2. Notice that it is similar to the case (4) in that the orientifold acts non-trivially on the three sphere. Noting that we expect symplectic gauge group, we find

$$Y = 2\left(\frac{8\pi^2}{g^2} - i\theta\right). \tag{7.5}$$

Then, with the identification of t as the glueball field S, (7.4) is indeed the Veneziano-Yankielowicz superpotential for the gauge group $\mathrm{Sp}(N)$, up to a power series in S.

If $N \leq -2$, at the point $(e^{-Y}, e^{-t}) = (\infty, 1)$ we have a large $S^3_<$ with $2\tilde{N}$ anti-D6-branes supporting an $\mathrm{Sp}(\tilde{N})$ gauge field. The relation of the parameter Y and the holomorphic gauge coupling is the same as (7.5) up to sign. With the identification $S = -t$, (7.4) agrees with the Veneziano-Yankielowicz superpotential for the $\mathrm{Sp}(\tilde{N})$ super Yang-Mills, since $-(N+1) = \tilde{N} + 1$ is the dual Coxeter number of $\mathrm{Sp}(\tilde{N})$.

One may also consider the case $N = -1$. In the resolved side, the superpotential from the flux is exactly canceled by the contribution form the crosscap instantons. The superpotential is simply $W = -Yt/2$ and $\partial_t W = 0$ requires $Y = 0$. Indeed, we do not have any candidate classical limit with deformed conifold for this value of the charge, as we have seen. What is most interesting is that the resolved branch has no singularity in the interior. By the combined effect of the flux and worldsheet instantons, the singularity, which was present in $\mathcal{N} = 2$ systems, is completely washed out!

Let us next consider the case of an anti-O6$^+$-plane and $2N > 4$ D6-branes on $S^3_>$ or $2\tilde{N} = -2N + 4 > 4$ anti-D6-branes on $S^3_<$. In this case, the crosscap contribution in (7.2) changes sign, and the equation for the moduli space is

$$e^{-Y} = (1 - e^{-t})^{2N-2}. \tag{7.6}$$

It is again a complex plane with three marked points. For $N > 2$, the holomorphic gauge coupling is related to the parameter Y by

$$Y = \frac{8\pi^2}{g^2} - i\theta\,, \tag{7.7}$$

and we have the expected behavior of the superpotential for $S = t/2$ near the point $(e^{-Y}, e^{-t}) = (0, 1)$, as $2(N-1)$ is the dual Coxeter number of the gauge group $\mathrm{SO}(2N)$. For $N < 0$, the relation of Y and the gauge coupling is opposite, but again the superpotential behaves as it should.

All these cases having been checked quite nicely, we briefly comment on the exceptional cases $N = 0, 1, 2$. Here we expect a μ-transition between vacua with $\mathrm{SO}(2N)$ and $\mathrm{SO}(4-2N)$ super Yang-Mills theories to be possible, but this will occur on a branch of moduli space that is not described by the exact superpotential. Though we do not have very powerful tools of analysis for the new branch, we can at least make a guess and check the consistency by examining the order of poles of holomorphic parameters at classical points. The expected branch structure is summarized in figure 15.

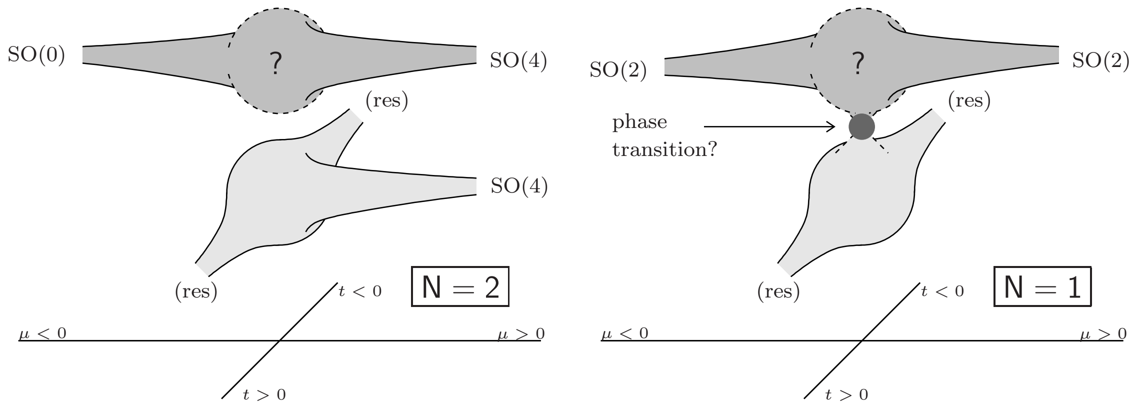

Figure 15: Expected structure of moduli space for case (2) with $N = 2$ and 1. The branch with darker shade contains vacua that are not accounted for by the superpotential.

For the case $N = 2$ (related to the case $N = 0$ by sign flip of Y), the superpotential accounts for only two vacua of the super Yang-Mills theory on large $S^3_>$. Other vacua should sit on a "new" branch which we expect to contain also the deformed vacuum with negative μ. The holomorphic parameter e^{-Y} has double zero at $\mu \to +\infty$, whereas the complex volume of minimal three-cycle at $\mu \to -\infty$ is minus $Y/2$ so that $e^{-Y/2}$ has a simple pole there. The new branch will therefore be a cylinder parametrized by $e^{-Y/2}$.

For $N = 1$, the flux and crosscap terms in the superpotential cancel out in the same way as the Sp case with $N = -1$. A branch of the moduli space is therefore a cylinder interpolating two resolved classical points. We expect another branch of vacua with low energy U(1) gauge dynamics that contains deformed classical points with either sign of μ. It would be interesting to study how the two branches are connected.

8. Conclusions

We have discussed supersymmetric quantum transitions between various orientifolds of the conifold. We have constructed the possible orientifolds of the deformed and resolved conifold in type-IIA string theory. In the primary case where this is possible, we have answered our basic question by considering the M-theory lift of the various IIA orientifold configurations. We identified the corresponding G_2 holonomy manifolds, and studied the quantum moduli space connecting different configurations through their topology and also the IIA exact superpotential.

Our main results are valid for the orientifold $z_i \to \bar{z}_i$, of deformed conifold $\sum_i z_i^2 = \mu$. With μ real, this has two phases. Depending on whether μ is positive or negative, the orientifold fixes the S^3, or acts freely so that the minimal cycle is an $\mathbb{RP}^3$. The transition between positive and negative μ is possible only for special values of RR charge. Once we fix the supersymmetry, one may either consider the O6$^\pm$-plane ($\mu > 0$) with charge ± 2 and increase the charge by adding D6-branes, or start with free orientifolds ($\mu < 0$) with zero charge and decrease the charge by adding anti-D6-branes. Therefore, μ can flip the sign only when the total charge $(N - 2)$ equals $0, -1$ or -2.

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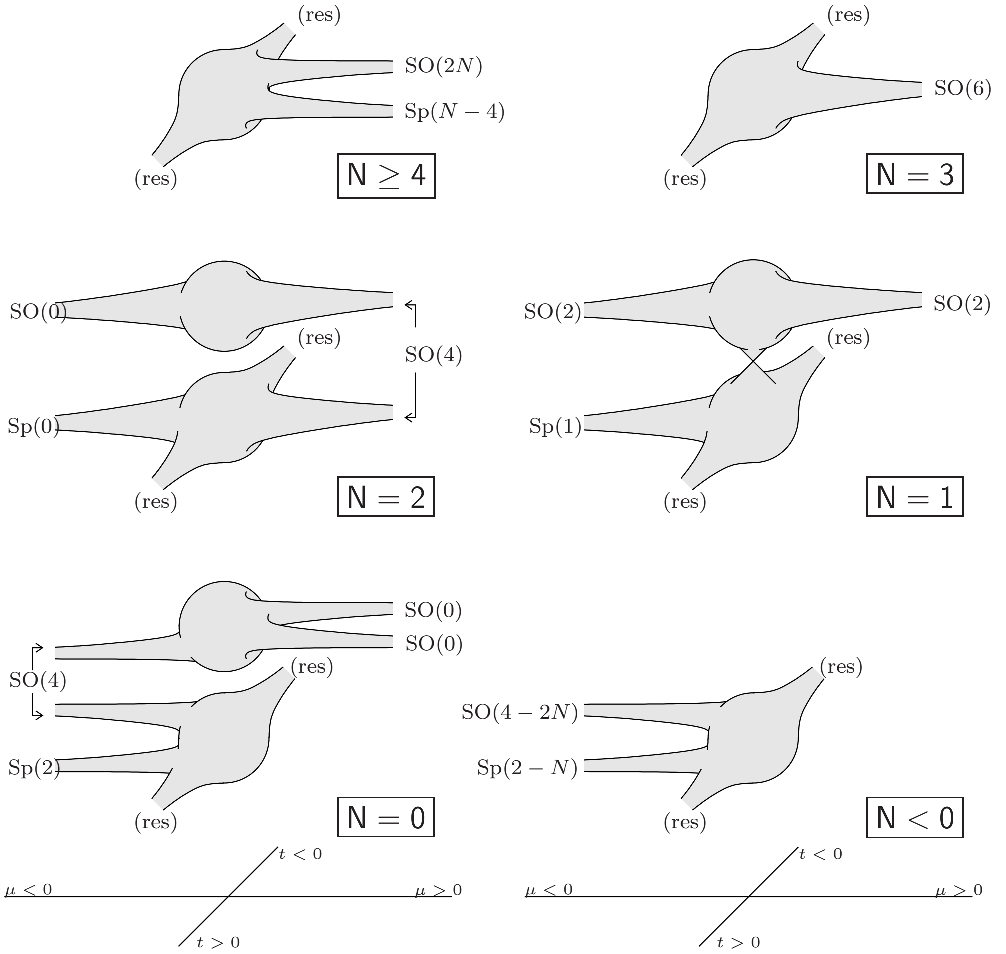

Figure 16: Moduli space of vacua for various total RR charge $(N-2)$.

On the other hand, some deformed or resolved geometries uplift to D_N-type orbifolds of the G_2 holonomy spaces $\mathbb{B}_7$ or $\mathbb{D}_7$, both of which are topologically $\mathbb{R}^4 \times S^3$. Remarkably, we found that this is true also for negatively large RR charges $(N-2 \leq -3)$, though the action of the dihedral group turned out to be non-standard. A careful analysis of the behavior of membrane instanton factors allows us to determine the structure of quantum moduli space unambiguously. Also, the exact IIA superpotential tells us how the resolved vacua are connected smoothly to other vacua.

We concluded that when $N = 0, 1, 2$ the moduli space consists of two branches. For $N = 0$ and $N = 2$, the two branches meet at infinity where there is a weakly coupled SO(4) super Yang-Mills theory, and each branch contains two of the four vacua. For $N = 1$, the branch of mass-gapped vacua meets the branch of vacua with infrared U(1) at a phase transition point, and we found a precise description of the transition via a mirror type-IIB picture.

The quantum moduli space of IIA orientifolded conifolds thus depends on the RR charge in an interesting way. We summarize these main results of our paper in figure 16. We found similar results in other cases of orientifolded conifolds as discussed in section 7.

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A. Conifold

Ricci-flat Kähler metrics for conifold or its small deformation or resolution were obtained in [30]. The singular conifold is defined by $\sum_{i=1}^4 z_i^2 = 0$ or equivalently by $\det W = 0$ with

$$W = \frac{1}{\sqrt{2}} \begin{pmatrix} z_1 + iz_2 & -z_3 + iz_4 \\ z_3 + iz_4 & z_1 - iz_2 \end{pmatrix}. \tag{A.1}$$

Regarding z_i as holomorphic coordinates and putting $K = \rho^{2/3}$ with $\rho \equiv \mathrm{Tr}\, WW^\dagger$, one obtains a Ricci-flat Kähler metric which is symmetric under $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ acting on W as $W \to LWR^\dagger$. To see the symmetry of the metric, we use the coordinates (which is slightly different from the one conventionally used)

$$W = X(\theta, \phi, \psi) \cdot W_0 \cdot \tilde{X}(\tilde{\theta}, \tilde{\phi}, \tilde{\psi})^\dagger , \qquad W_0 = r^{3/2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{A.2}$$

or more explicitly

$$W = r^{3/2} \begin{pmatrix} \cos\frac{\theta}{2} \cos\frac{\tilde{\theta}}{2} e^{\frac{i}{2}(\psi - \tilde{\psi} + \phi - \tilde{\phi})} & \cos\frac{\theta}{2} \sin\frac{\tilde{\theta}}{2} e^{\frac{i}{2}(\psi - \tilde{\psi} + \phi + \tilde{\phi})} \\ \sin\frac{\theta}{2} \cos\frac{\tilde{\theta}}{2} e^{\frac{i}{2}(\psi - \tilde{\psi} - \phi - \tilde{\phi})} & \sin\frac{\theta}{2} \sin\frac{\tilde{\theta}}{2} e^{\frac{i}{2}(\psi - \tilde{\psi} - \phi + \tilde{\phi})} \end{pmatrix} \tag{A.3}$$

and get

$$\begin{aligned} ds^2 &= dr^2 + r^2 ds^2_{T^{1,1}} , \\ ds^2_{T^{1,1}} &= \frac{1}{6}(\sigma_1^2 + \sigma_2^2 + \tilde{\sigma}_1^2 + \tilde{\sigma}_2^2) + \frac{1}{9}(\sigma_3 - \tilde{\sigma}_3)^2 \\ &= \frac{1}{6}(d\theta^2 + \sin^2\theta d\phi^2 + d\tilde{\theta}^2 + \sin^2\tilde{\theta} d\tilde{\phi}^2) + \\ &\quad + \frac{1}{9}(d(\psi - \tilde{\psi}) + \cos\theta d\phi - \cos\tilde{\theta} d\tilde{\phi})^2. \end{aligned} \tag{A.4}$$

The coordinate $\check{\psi} \equiv \psi - \tilde{\psi}$ has the period 4π and defines an S^1 which is fibered over $S^2 \times S^2$ labeled by (θ, ϕ) and $(\tilde{\theta}, \tilde{\phi})$. This metric has an $\mathrm{O}(4) \times \mathrm{U}(1)$ symmetry: $\mathrm{SO}(4) \simeq (\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R)/\mathbb{Z}_2$ acts on W as explained and in particular z_i are transformed as a 4-vector. A parity transform in O(4) exchanges the two S^2's, and the U(1) shifts ψ or acts on z_i as phase rotation.

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Small resolution is an $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ bundle over $\mathbb{CP}^1$. To write down the Kähler metric, write the matrix W with vanishing determinant as

$$W = \begin{pmatrix} -u\lambda & u \\ -y\lambda & y \end{pmatrix} = \begin{pmatrix} x & -x\mu \\ v & -v\mu \end{pmatrix}. \tag{A.5}$$

(λ, μ) with $\lambda\mu = 1$ are regarded as coordinates on $\mathbb{CP}^1$, and from the relation between (u, y) and (x, v) one finds they are coordinates on the fiber. A natural ansatz for the Kähler potential of the resolved conifold is

$$K = K(\rho) + 4a^2 \ln(1 + |\lambda|^2), \tag{A.6}$$

the second term yielding a Fubini-Study metric on $\mathbb{CP}^1$. From the Ricci-flatness one finds that $r^2 \equiv \rho \frac{dK}{d\rho}$ has to satisfy

$$r^4(r^2 + 6a^2) = c\rho^2 + c' \tag{A.7}$$

for some constants c, c'. Setting $c = 1, c' = 0$ one obtains

$$ds^2 \propto k^{-1}(r)dr^2 + \frac{r^2}{6}(\sigma_1^2 + \sigma_2^2) + \frac{r^2 + 4a^2}{6}(\tilde{\sigma}_1^2 + \tilde{\sigma}_2^2) + \frac{r^2 k(r)}{9}(\sigma_3 - \tilde{\sigma}_3)^2\,, \tag{A.8}$$

with $k(r) = \frac{r^2+6a^2}{r^2+4a^2}$. The metric is invariant under $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)$, but the $\mathbb{Z}_2$ symmetry of singular conifold is lost.

Small deformation is defined by $\sum_i z_i^2 = \epsilon^2$ or $\det W = \frac{\epsilon^2}{2}$. Hereafter we assume ϵ to be real positive, as the Ricci-flat metric will depend only on the modulus $|\epsilon|^2$. The $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ invariant metric can be found by assuming the Kähler potential to be a function $K(\rho)$ of $\rho = \mathrm{Tr}\, WW^\dagger$. The Ricci-flatness can be solved easily by introducing $\rho = \epsilon^2 \cosh\tau$ and putting

$$W_0 = \frac{\epsilon}{\sqrt{2}} \begin{pmatrix} e^{\tau/2} & 0 \\ 0 & e^{-\tau/2} \end{pmatrix}. \tag{A.9}$$

The Ricci-flatness amounts to $\frac{dK}{d\rho} = \frac{(\sinh 2\tau - 2\tau)^{1/3}}{\sinh\tau} \equiv k(\tau)$, and one obtains the following metric

$$ds^2 \propto k(\tau) \left\{ \frac{4}{3k(\tau)^3}(d\tau^2 + (\sigma_3 - \tilde{\sigma}_3)^2) + \cosh\tau(\sigma_1^2 + \tilde{\sigma}_1^2 + \sigma_2^2 + \tilde{\sigma}_2^2) - 2(\sigma_1\tilde{\sigma}_1 + \sigma_2\tilde{\sigma}_2) \right\}. \tag{A.10}$$

The metric is invariant under O(4) but not under U(1).

B. Properties of $Li_2(z)$

We can define the Euler dilogarithm function[6] in the disk $|z| < 1$ as a convergent power series:[7]

$$Li_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}\,. \tag{B.1}$$

[6] Defined by Euler in 1768.

[7] For more properties of this function see [38].

The function can be extended to the whole complex plane as a multi valued analytical function. There is a branch cut from $z = 1$ to $z = \infty$. Alternatively it can be defined as an integral:

$$Li_2(z) = -\int_0^z \frac{\log(1-z)}{z} dz\,. \tag{B.2}$$

A related function often found in literature is the Rogers L-function:

$$L(z) = Li_2(z) + \frac{1}{2}\log z \log(1-z)\,. \tag{B.3}$$

Some functional equations that we will use in the main text are:

- The Euler identity provides an expansion around the branch point $z = 1$:

$$Li_2(1-z) = -Li_2(z) + \frac{\pi^2}{6} - \log z \log(1-z)\,. \tag{B.4}$$

  Or in terms of the Rogers L-function:

$$L(z) + L(1-z) = L(1)\,. \tag{B.5}$$

- The expansion around $z = \infty$

$$Li_2\left(\frac{1}{z}\right) = -Li_2(z) - \frac{\pi^2}{6} + \frac{1}{2}(\log(-z))^2\,. \tag{B.6}$$

- A simple relation between the value of $Li_2(z)$ and $Li_2(z^2)$

$$Li_2(z) + Li_2(-z) = \frac{1}{2}Li_2(z^2)\,. \tag{B.7}$$

- The above relations can be obtained in terms of the Abel identity

$$\begin{aligned} Li_2(x) + Li_2(y) &= Li_2(xy) + Li_2\left(\frac{x(1-y)}{1-xy}\right) + \\ &\quad + Li_2\left(\frac{y(1-x)}{1-xy}\right) + \log\left(\frac{1-x}{1-xy}\right)\log\left(\frac{1-y}{1-xy}\right) \end{aligned} \tag{B.8}$$

  Or in terms of the Rogers L-function:

$$L(x) + L(y) = L(xy) + L\left(\frac{x(1-y)}{1-xy}\right) + L\left(\frac{y(1-x)}{1-xy}\right)\,. \tag{B.9}$$

And some particular values of the dilogarithm:

$$Li_2(1) = \frac{\pi^2}{6}\,, \qquad Li_2(-1) = -\frac{\pi^2}{12}\,, \qquad Li_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{1}{2}(\log(2))^2\,, \tag{B.10}$$

$$L(1) = \frac{\pi^2}{6}\,, \qquad L(0) = 0\,. \tag{B.11}$$

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